Final Exam
(Physics 230A, Spring 2006)
Due 5PM, June 9, 2006

1. (30 pts) In the class we argued that in relativitistc system particle numbers are not conserved, therefore we need to use the quantum field theory to describe the system. However, even a non-relativistic system can be second quantized, and it has many applications in condense matter physics and statistical mechanics. In this problem we consider a non-relativistic quantum field and derive its relation with the usual quantum mechanics. A free non-relativistic quantum field is described by the Lagrangian density ($\hbar = 1$)

$$\mathcal{L}(\vec{x}, t) = i\psi^\dagger(\vec{x}, t)\frac{\partial}{\partial t}\psi(\vec{x}, t) - \frac{1}{2m}\left(\nabla\psi^\dagger(\vec{x}, t)\right) \cdot (\nabla\psi(\vec{x}, t)). \tag{1}$$

(a) (5 pts) Find the conjugate momentum density $\pi(\vec{x}, t)$ of $\psi(\vec{x}, t)$ and the equation of motion for $\psi(\vec{x}, t)$.
(b) (5 pts) Expand $\psi(\vec{x}, t)$ and the conjugate momentum density in terms of a complete set of the solutions to the equation of motion for a free field (assuming $\psi(\vec{x}, t)$ in a finite box of volume $V = L^3$ as in the lecture notes). Specify the relation between the energy and momentum. (Is there any negative energy solution?) Now impose the standard commutation relation between ψ and π for the canonical quantization. Show that the coefficients (now promoted to operators) of each momentum mode (with appropriate normalization) satisfy the commutation algebra of the creation and annihilation operators of the harmonic oscillator.
(c) (5 pts) Define the position operator and the momentum operator as,

$$\vec{X}(t) = \int d^3x\psi^\dagger(\vec{x}, t)\vec{x}\psi(\vec{x}, t), \quad \vec{P}(t) = \int d^3x\psi^\dagger(\vec{x}, t)(-i\nabla)\psi(\vec{x}, t). \tag{2}$$

Show that $\vec{X}(t)$ and $\vec{P}(t)$ satisfy the Heisenberg's commutation relation, $[X_i(t), P_j(t)] = i\delta_{ij}$
(d) (5 pts) In non-relativistic quantum mechanics, the Fermi or Bose statistics of a particle is put in by hand. Show that if we impose the anticommutation relation on ψ and its conjugate momentum instead of the commutation relation, $\vec{X}(t)$ and $\vec{P}(t)$ still satisfy the same Heisenberg's commutation relation, $[X_i(t), P_j(t)] = i\delta_{ij}$.
(e) (5 pts) Consider a state $|\Phi(t)\rangle = \int d^3x\phi(\vec{x}, t)\psi^\dagger(\vec{x}, 0)|0\rangle$, which is a linear superposition of the position eigenstates at $t = 0$ with a c-number function $\phi(\vec{x}, t)$ ($\phi(\vec{x}, t)$ is not an operator). The state $|\Phi(t)\rangle$ obeys the equation of motion,

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H(0)|\Phi(t)\rangle, \tag{3}$$

where $H = \int d^3x\mathcal{H}(\vec{x}, t)$ with $\mathcal{H}(\vec{x}, t) = (\nabla\psi^\dagger \cdot \nabla\psi)/(2m) + V(\vec{x})\psi^\dagger\psi$, and we have included a time-independent potential $V(\vec{x})$. Show that the c-number function $\phi(\vec{x}, t)$ obeys the Schrödinger's equation of quantum mechanics.
(f) (5 pts) Construct a state of two identical particles with a superposition function $\phi(\vec{x}, \vec{y}, s, s', t)$, where s and s' are z-components of the particle spins:

$$|\Phi(t)\rangle = \sum_{s,s'}\int d^3x\int d^3y\phi(\vec{x}, \vec{y}, s, s', t)\psi_s^\dagger(\vec{x}, 0)\psi_{s'}^\dagger(\vec{y}, 0)|0\rangle. \tag{4}$$

Show that if the quantum field obeys the commutation/anticommutation relations, only the symmetric/antisymmetric part of the function ϕ under $(\vec{x}, s) \leftrightarrow (\vec{y}, s')$ is relevant, and therefore the wavefunction of two identical bosons/fermions need to be symmetric/antisymmetric, *i.e.*, we should choose

$$\phi(\vec{y}, \vec{x}, s', s, t) = \pm\phi(\vec{x}, \vec{y}, s, s', t). \tag{5}$$

2. (10 pts) Prove the following identity,

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'_\nu - p_\nu)}{2m}\right]u(p), \tag{6}$$

where $u(p)$ is a positive energy spinor of momentum p and mass m.

3. (10 pts) Let us consider a somewhat unusual gauge fixing condition (called axial gauge) for the electromagnetic field. The gauge-fixed Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\eta_\mu A^\mu)^2, \tag{7}$$

where η_μ is a constant 4-vector. Find the causal Green's function (Feynman propagator) $D_F^{\mu\nu}(x-y)[= -i\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle]$ in this gauge. It will of course depend on η_μ. (Hint: It is easier to find its Fourier transformation $\tilde{D}_F^{\mu\nu}(k)$ first.)

4. (20 pts) Consider the Lagrangian density

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m_N)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\pi^2\phi^2 + i\lambda\overline{\psi}\gamma_5\psi\phi. \tag{8}$$

It can be thought of as describing the interactions between the nucleon (fermion ψ) and the neutral pion (real scalar ϕ) with a coupling λ.

(a) (10 pts) Draw the tree-level $[\mathcal{O}(\lambda^2)]$ Feynman diagrams for the scattering of a nucleon and an anti-nucleon, $N(p,r) + \overline{N}(k,s) \to N(p',r') + \overline{N}(k',s')$, where p, k and p', k' are incoming and outgoing momenta and r, s and r', s' are the helicities of the incoming and outgoing particles respectively. Label the diagrams appropriately (including the arrows, external states, vertices and the internal propagators, representing a scalar propagator by a dashed line).

(b) (10 pts) Find the (tree-level) amplitude for the scattering process in (a), $\langle\overline{N}(k',s'), N(p',r')|S|N(p,r), \overline{N}(k,s)\rangle$. Be careful about the relative sign between different diagrams. Argue or show why the relative sign is what it is.

5. (10 pts) Muon is a particle just like the electron, except that it is heavier ($m_\mu \approx 200 m_e$). Its interaction with the electromagnetic field is exactly the same as the interactions of the electron. Draw all the possible Feynman diagrams which contribute to the process $e^+ + e^- \to e^+ + e^- + \mu^+ + \mu^-$ at the tree-level $[\mathcal{O}(e^4)]$ in quantum electrodynamics.