4 Dirac Equation

To solve the negative probability density problem of the Klein-Gordon equation, people were looking for an equation which is first order in $\partial/\partial t$. Such an equation is found by Dirac.

It is difficult to take the square root of $-\hbar^2c^2\nabla^2 + m^2c^4$ for a single wave function. One can take the inspiration from E&M: Maxwell's equations are first-order but combining them gives the second order wave equations.

Imagining that ψ consists of N components ψ_l,

$$\frac{1}{c}\frac{\partial \psi_l}{\partial t} + \sum_{k=1}^{3}\sum_{n=1}^{N}\alpha^k_{ln}\frac{\partial \psi_n}{\partial x^k} + \frac{imc}{\hbar}\sum_{n=1}^{N}\beta_{ln}\psi_n = 0, \tag{53}$$

where $l = 1, 2, \ldots, N$, and $x^k = x,\ y,\ z,\ k = 1\,, 2,\ 3$.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix}, \tag{54}$$

and α^k, β are $N \times N$ matrices. Using the matrix notation, we can write the equations as

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \frac{imc}{\hbar}\beta\psi = 0, \tag{55}$$

where $\boldsymbol{\alpha} = \alpha^1\hat{\boldsymbol{x}} + \alpha^2\hat{\boldsymbol{y}} + \alpha^3\hat{\boldsymbol{z}}$. N components of ψ describe a new degree of freedom just as the components of the Maxwell field describe the polarization of the light quantum. In this case, the new degree of freedom is the spin of the particle and ψ is called a spinor.

We would like to have positive-definite and conserved probability, $\rho = \psi^\dagger\psi$, where $\psi^\dagger$ is the hermitian conjugate of ψ (so is a row matrix). Taking the hermitian conjugate of Eq. (55),

$$\frac{1}{c}\frac{\partial \psi^\dagger}{\partial t} + \boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha} - \frac{imc}{\hbar}\psi^\dagger\beta^\dagger = 0. \tag{56}$$

Multiplying the above equation by ψ and then adding it to $\psi^\dagger\times$ (55), we obtain

$$\frac{1}{c}\left(\psi^\dagger\frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t}\psi\right) + \boldsymbol{\nabla}\psi^\dagger\cdot\boldsymbol{\alpha}^\dagger\psi + \psi^\dagger\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}\psi + \frac{imc}{\hbar}(\psi^\dagger\beta\psi - \psi^\dagger\beta^\dagger\psi) = 0. \tag{57}$$

The continuity equation

$$\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \tag{58}$$

can be obtained if $\boldsymbol{\alpha}^\dagger = \boldsymbol{\alpha}$, $\beta^\dagger = \beta$, then

$$\frac{1}{c}\frac{\partial}{\partial t}(\psi^\dagger\psi) + \boldsymbol{\nabla}\cdot(\psi^\dagger\boldsymbol{\alpha}\psi) = 0 \tag{59}$$

with

$$\boldsymbol{j} = c\psi^\dagger\boldsymbol{\alpha}\psi. \tag{60}$$

From Eq. (55) we can obtain the Hamiltonian,

$$H\psi = i\hbar\frac{\partial\psi}{\partial t} = \left(c\boldsymbol{\nabla}\cdot\frac{\hbar}{i}\boldsymbol{\nabla} + \beta mc^2\right)\psi. \tag{61}$$

One can see that H is hermitian if $\boldsymbol{\alpha}$, β are hermitian.

To derive properties of $\boldsymbol{\alpha}$, β, we multiply Eq. (55) by the conjugate operator,

$$\begin{aligned}\left(\frac{1}{c}\frac{\partial}{\partial t} - \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \frac{imc}{\hbar}\beta\right)\left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{imc}{\hbar}\beta\right)\psi = 0 \\ \Rightarrow \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \alpha^i\alpha^j\partial_i\partial_j + \frac{m^2c^2}{\hbar^2}\beta^2 - \frac{imc}{\hbar}(\beta\alpha^i + \alpha^i\beta)\partial_i\right]\psi = 0\end{aligned} \tag{62}$$

We can rewrite $\alpha^i\alpha^j\partial_i\partial_j$ as $\frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\partial_i\partial_j$. Since it's a relativistic system, the second order equation should coincide with the Klein-Gordon equation. Therefore, we must have

$$\alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}I \tag{63}$$

$$\beta\alpha^i + \alpha^i\beta = 0 \tag{64}$$

$$\beta^2 = I \tag{65}$$

Because

$$\beta\alpha^i = -\alpha^i\beta = (-I)\alpha^i\beta, \tag{66}$$

if we take the determinant of the above equation,

$$\det\beta\det\alpha^i = (-1)^N\det\alpha^i\det\beta, \tag{67}$$

we find that N must be even. Next, we can rewrite the relation as

$$(\alpha^i)^{-1}\beta\alpha^i = -\beta \quad \text{(no summation)}. \tag{68}$$

Taking the trace,

$$\mathrm{Tr}\left[(\alpha^i)^{-1}\beta\alpha^i\right] = \mathrm{Tr}\left[(\alpha^i\alpha^i)^{-1}\beta\right] = \mathrm{Tr}[\beta] = \mathrm{Tr}[-\beta], \tag{69}$$

we obtain $\mathrm{Tr}[\beta] = 0$. Similarly, one can derive $\mathrm{Tr}[\alpha^i] = 0$.

Covariant form of the Dirac equation

Define

$$\begin{aligned}
\gamma^0 &= \beta, \\
\gamma^j &= \beta\alpha^j, \quad j = 1, 2, 3 \\
\gamma^\mu &= (\gamma^0, \gamma^1, \gamma^2, \gamma^3), \quad \gamma_\mu = g_{\mu\nu}\gamma^\nu
\end{aligned} \tag{70}$$

Multiply Eq. (55) by $i\beta$,

$$\begin{aligned}
i\beta \times \left(\frac{1}{c}\frac{\partial}{\partial t} + \boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \frac{imc}{\hbar}\beta\right)\psi &= 0 \\
\Rightarrow \left(i\gamma^0\frac{\partial}{\partial x^0} + i\gamma^j\frac{\partial}{\partial x^j} - \frac{mc}{\hbar}\right)\psi &= \left(i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right)\psi = 0
\end{aligned} \tag{71}$$

Using the short-hand notation: $\gamma^\mu\partial_\mu \equiv \not{\partial}$, $\gamma^\mu A_\mu \equiv \not{A}$,

$$\left(i\not{\partial} - \frac{mc}{\hbar}\right)\psi = 0 \tag{72}$$

From the properties of the α^j and β matrices, we can derive

$$\gamma^{0\dagger} = \gamma^0, \quad \text{(hermitian)} \tag{73}$$

$$\gamma^{j\dagger} = (\beta\alpha^j)^\dagger = \alpha^{j\dagger}\beta^\dagger = \alpha^j\beta = -\beta\alpha^j = -\gamma^j, \quad \text{(anti-hermitian)} \tag{74}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \tag{75}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I. \quad \text{(Clifford algebra).} \tag{76}$$

Conjugate of the Dirac equation is given by

$$\begin{aligned}
-i\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - \frac{mc}{\hbar}\psi^\dagger &= 0 \\
\Rightarrow -i\partial_\mu\psi^\dagger\gamma^0\gamma^\mu\gamma^0 - \frac{mc}{\hbar}\psi^\dagger &= 0
\end{aligned} \tag{77}$$

We will define the Dirac adjoint spinor $\overline{\psi}$ by $\overline{\psi} \equiv \psi^\dagger\gamma^0$. Then

$$i\partial_\mu\overline{\psi}\gamma^\mu + \frac{mc}{\hbar}\overline{\psi} = 0. \tag{78}$$

The four-current is

$$\frac{j^\mu}{c} = \overline{\psi}\gamma^\mu\psi = \left(\rho, \frac{\boldsymbol{j}}{c}\right), \quad \partial_\mu j^\mu = 0. \tag{79}$$

Properties of the γ^μ matrices

We may form new matrices by multiplying γ matrices together. Because different γ matrices anticommute, we only need to consider products of different γ's and the order is not important. We can combine them in 2^4-1 ways. Plus the identity we have 16 different matrices,

$$\begin{aligned} & I \\ & \gamma^0,\ i\gamma^1,\ i\gamma^2, i\gamma^3 \\ & \gamma^0\gamma^1,\ \gamma^0\gamma^2,\ \gamma^0\gamma^3,\ i\gamma^2\gamma^3,\ i\gamma^3\gamma^1,\ i\gamma^1\gamma^2 \\ & i\gamma^0\gamma^2\gamma^3,\ i\gamma^0\gamma^3\gamma^1,\ i\gamma^0\gamma^1\gamma^2,\ \gamma^1\gamma^2\gamma^3 \\ & i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5 (= \gamma^5). \end{aligned} \tag{80}$$

Denoting them by Γ_l, $l = 1, 2, \cdots, 16$, we can derive the following relations.

(a) $\Gamma_l\Gamma_m = a_{lm}\Gamma_n,\ a_{lm} = \pm 1$ or $\pm i$.

(b) $\Gamma_l\Gamma_m = I$ if and only if $l = m$.

(c) $\Gamma_l\Gamma_m = \pm\Gamma_m\Gamma_l$.

(d) If $\Gamma_l \neq I$, there always exists a Γ_k, such that $\Gamma_k\Gamma_l\Gamma_k = -\Gamma_l$.

(e) $\mathrm{Tr}(\Gamma_l) = 0$ for $\Gamma_l \neq I$.
Proof:

$$\mathrm{Tr}(-\Gamma_l) = \mathrm{Tr}(\Gamma_k\Gamma_l\Gamma_k) = \mathrm{Tr}(\Gamma_l\Gamma_k\Gamma_k) = \mathrm{Tr}(\Gamma_l)$$

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(f) Γ_l are linearly independent: $\sum_{k=1}^{16} x_k\Gamma_k = 0$ only if $x_k = 0,\ k = 1, 2, \cdots, 16$.
Proof:

$$\left(\sum_{k=1}^{16} x_k\Gamma_k\right)\Gamma_m = x_m I + \sum_{k\neq m} x_k\Gamma_k\Gamma_m = x_m I + \sum_{k\neq m} x_k a_{km}\Gamma_n = 0\ (\Gamma_n \neq I).$$

Taking the trace, $x_m\mathrm{Tr}(I) = -\sum_{k\neq m} x_k a_{km}\mathrm{Tr}(\Gamma_n) = 0 \Rightarrow x_m = 0.$ for any m. This implies that Γ_k's cannot be represented by matrices smaller than 4×4. In fact, the smallest representations of Γ_k's are 4×4 matrices. (Note that this 4 is not the dimension of the space-time. the equality is accidental.)

(g) Corollary: any 4×4 matrix X can be written uniquely as a linear combination of the Γ_k's.

$$\begin{aligned} X &= \sum_{k=1}^{16} x_k\Gamma_k \\ \mathrm{Tr}(X\Gamma_m) &= x_m\mathrm{Tr}(\Gamma_m\Gamma_m) + \sum_{k\neq m} x_k\mathrm{Tr}(\Gamma_k\Gamma_m) = x_m\mathrm{Tr}(I) = 4x_m \\ x_m &= \frac{1}{4}\mathrm{Tr}(x\Gamma_m) \end{aligned}$$

(h) Stronger corollary: $\Gamma_l \Gamma_m = a_{lm} \Gamma_n$ where Γ_n is a different Γ_n for each m, given a fixed l.
Proof: If it were not true and one can find two different Γ_m, $\Gamma_{m'}$ such that $\Gamma_l \Gamma_m = a_{lm} \Gamma_n$, $\Gamma_l \Gamma_{m'} = a_{lm'} \Gamma_n$, then we have

$$\Gamma_m = a_{lm} \Gamma_l \Gamma_n, \ \Gamma_{m'} = a_{lm'} \Gamma_l \Gamma_n \ \Rightarrow \Gamma_m = \frac{a_{lm}}{a_{lm'}} \Gamma_{m'},$$

which contradicts that γ_k's are linearly independent.

(i) Any matrix X that commutes with γ^μ (for all μ) is a multiple of the identity.
Proof: Assume X is not a multiple of the identity. If X commutes with all γ^μ then it commutes with all Γ_l's, i.e., $X = \Gamma_l X \Gamma_l$. We can express X in terms of the Γ matrices,

$$X = x_m \Gamma_m + \sum_{k \neq m} x_k \Gamma_k, \ \Gamma_m \neq I.$$

There exists a Γ_i such that $\Gamma_i \Gamma_m \Gamma_i = -\Gamma_m$. By the hypothesis that X commutes with this Γ_i, we have

$$\begin{aligned} X &= x_m \Gamma_m + \sum_{k \neq m} x_k \Gamma_k = \Gamma_i X \Gamma_i \\ &= x_m \Gamma_i \Gamma_m \Gamma_i + \sum_{k \neq m} x_k \Gamma_i \Gamma_k \Gamma_i \\ &= -x_m \Gamma_m + \sum_{k \neq m} \pm x_k \Gamma_k. \end{aligned}$$

Since the expansion is unique, we must have $x_m = -x_m$. Γ_m was arbitrary except that $\Gamma_m \neq I$. This implies that all $x_m = 0$ for $\Gamma_m \neq I$ and hence $X = aI$.

(j) Pauli's fundamental theorem: Given two sets of 4×4 matrices γ^μ and γ'^μ which both satisfy

$$\{\gamma^\mu, \ \gamma^\nu\} = 2 g^{\mu\nu} I,$$

there exists a nonsingular matrix S such that

$$\gamma'^\mu = S \gamma^\mu S^{-1}.$$

Proof: F is an arbitrary 4×4 matrix, set Γ_i is constructed from γ^μ and Γ'_i is constructed from γ'^μ. Let

$$S = \sum_{i=1}^{16} \Gamma'_i F \Gamma_i.$$

$$\begin{aligned} &\Gamma_i \Gamma_j = a_{ij} \Gamma_k \\ &\Gamma_i \Gamma_j \Gamma_i \Gamma_j = a_{ij}^2 \Gamma_k^2 = a_{ij}^2 \\ &\Gamma_i \Gamma_i \Gamma_j \Gamma_i \Gamma_j \Gamma_j = \Gamma_j \Gamma_i = a_{ij}^2 \Gamma_i \Gamma_j = a_{ij}^3 \Gamma_k \\ &\Gamma'_i \Gamma'_j = a_{ij} \Gamma'_k \end{aligned}$$

For any i,

$$\Gamma_i' S \Gamma_i = \sum_j \Gamma_i' \gamma_j' F \Gamma_j \Gamma_i = \sum_j a_{ij}^4 \Gamma_k' F \Gamma_k = \sum_j \Gamma_k' F \Gamma_k = S, \ (a_{ij}^4 = 1).$$

It remains only to prove that S is nonsingular.

$$S' = \sum_{i=1}^{16} \Gamma_i G \Gamma_i', \quad \text{for } G \text{ arbitrary.}$$

By the same argument, we have $S' = \Gamma_i S' \Gamma_i'$.

$$S'S = \Gamma_i S' \Gamma_i' \Gamma_i' S \Gamma_i = \Gamma_i S' S \Gamma_i,$$

$S'S$ commutes with Γ_i for any i so $S'S = aI$. We can choose $a \neq 0$ because F, G are arbitrary, then S is nonsingular. Also, S is unique up to a constant. Otherwise if we had $S_1 \gamma^\mu S_1^{-1} = S_2 \gamma^\mu S_2^{-1}$, then $S_2^{-1} S_1 \gamma^\mu = \gamma^\mu S_2^{-1} S_1 \Rightarrow S_2^{-1} S_1 = aI$.

Specific representations of the γ^μ matrices

Recall $H = (-c\boldsymbol{\alpha}(i\hbar)\boldsymbol{\nabla} + \beta mc^2)$. In the non-relativistic limit, mc^2 term dominates the total energy, so it's convenient to represent $\beta = \gamma^0$ by a diagonal matrix. Recall $\mathrm{Tr}\beta = 0$ and $\beta^2 = I$, so we choose

$$\beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{81}$$

α^k's anticommute with β and are hermitian,

$$\alpha^k = \begin{pmatrix} 0 & A^k \\ (A^k)^\dagger & 0 \end{pmatrix}, \tag{82}$$

A^k: 2×2 matrices, anticommute with each other. These properties are satisfied by the Pauli matrices, so we have

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{83}$$

From these we obtain

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{84}$$

This is the "Pauli-Dirac" representation of the γ^μ matrices. It's most useful for system with small kinetic energy, *e.g.,* atomic physics.

Let's consider the simplest possible problem: free particle at rest. ψ is a 4-component wave-function with each component satisfying the Klein-Gordon equation,

$$\psi = \chi\, e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et)}, \tag{85}$$

where χ is a 4-component spinor and $E^2 = p^2c^2 + m^2c^4$.

Free particle at rest: $\boldsymbol{p} = 0$, ψ is independent of $\boldsymbol{x}$,

$$H\psi = (-i\hbar c\boldsymbol{\alpha}\cdot\nabla + mc^2\gamma^0)\psi = mc^2\gamma^0\psi = E\psi. \tag{86}$$

In Pauli-Dirac representation,$\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$, the 4 fundamental solutions are

$$\begin{aligned}
\chi_1 &= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad E = mc^2,\\
\chi_2 &= \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad E = mc^2,\\
\chi_3 &= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad E = -mc^2,\\
\chi_4 &= \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad E = -mc^2.
\end{aligned}$$

As we shall see, Dirac wavefunction describes a particle pf spin-1/2. χ_1, χ_2 represent spin-up and spin-down respectively with $E = mc^2$. χ_3, χ_4 represent spin-up and spin-down respectively with $E = -mc^2$. As in Klein-Gordon equation, we have negative energy solutions and they can not be discarded.

For ultra-relativistic problems (most of this course), the "Weyl" representation is more convenient.

$$\psi_{\mathrm{PD}} = \begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix} = \begin{pmatrix}\psi_A\\\psi_B\end{pmatrix}, \quad \psi_A = \begin{pmatrix}\psi_1\\\psi_2\end{pmatrix}, \ \psi_b = \begin{pmatrix}\psi_3\\\psi_4\end{pmatrix}. \tag{87}$$

In terms of ψ_A and ψ_B, the Dirac equation is

$$\begin{aligned}
i\frac{\partial}{\partial x^0}\psi_A + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_B &= \frac{mc}{\hbar}\psi_A,\\
-i\frac{\partial}{\partial x^0}\psi_B - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\psi_A &= \frac{mc}{\hbar}\psi_B.
\end{aligned} \tag{88}$$

Let's define

$$\psi_A = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2), \quad \psi_B = \frac{1}{\sqrt{2}}(\phi_2 - \phi_1) \tag{89}$$

and rewrite the Dirac equation in terms of ϕ_1 and ϕ_2,

$$\begin{aligned} i\frac{\partial}{\partial x^0}\phi_1 - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi_1 &= \frac{mc}{\hbar}\phi_2, \\ i\frac{\partial}{\partial x^0}\phi_2 + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi_2 &= \frac{mc}{\hbar}\phi_1. \end{aligned} \tag{90}$$

On can see that ϕ_1 and ϕ_2 are coupled only via the mass term. In ultra-relativistic limit (or for nearly massless particle such as neutrinos), rest mass is negligible, then ϕ_1 and ϕ_2 decouple,

$$\begin{aligned} i\frac{\partial}{\partial x^0}\phi_1 - i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi_1 &= 0, \\ i\frac{\partial}{\partial x^0}\phi_2 + i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi_2 &= 0, \end{aligned} \tag{91}$$

The 4-component wavefunction in the Weyl representation is written as

$$\psi_{\text{Weyl}} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{92}$$

Let's imagine that a massless spin-1/2 neutrino is described by ϕ_1, a plane wave state of a definite momentum $\boldsymbol{p}$ with energy $E = |\boldsymbol{p}|c$,

$$\phi_1 \propto e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x}-Et)}. \tag{93}$$

$$\begin{aligned} i\frac{\partial}{\partial x^0}\phi_1 &= i\frac{1}{c}\frac{\partial}{\partial t}\phi_1 = \frac{E}{\hbar c}\phi_1, \\ i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi_1 &= -\frac{1}{\hbar}\boldsymbol{\sigma}\cdot\boldsymbol{p}\phi_1 \\ \Rightarrow E\phi_1 &= |\boldsymbol{p}|c\phi_1 = -c\boldsymbol{\sigma}\cdot\boldsymbol{p}\phi_1 \quad \text{or} \quad \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\phi_1 = -\phi_1. \end{aligned} \tag{94}$$

The operator $h = \boldsymbol{\sigma}\cdot\boldsymbol{p}/|\boldsymbol{p}|$ is called the "helicity." Physically it refers to the component of spin in the direction of motion. ϕ_1 describes a neutrino with helicity -1 ("left-handed"). Similarly,

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\phi_2 = \phi_2, \quad (h = +1, \text{ "right-handed"}). \tag{95}$$

The γ^μ's in the Weyl representation are

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{96}$$

Exercise: Find the S matrix which transform between the Pauli-Dirac representation and the Weyl representation and verify that the γ^μ matrices in the Weyl representation are correct.