5 Lorentz Covariance of the Dirac Equation

We will set $\hbar = c = 1$ from now on.

In E&M, we write down Maxwell's equations in a given inertial frame, $\boldsymbol{x}$, t, with the electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$. Maxwell's equations are covariant with respecct to Lorentz transformations, i.e., in a new Lorentz frame, $\boldsymbol{x}'$, t', the equations have the same form, but the fields $\boldsymbol{E}'(\boldsymbol{x}', t')$, $\boldsymbol{B}'(\boldsymbol{x}', t')$ are different.

Similarly, Dirac equation is Lorentz covariant, but the wavefunction will change when we make a Lorentz transformation. Consider a frame F with an observer O and coordinates x^μ. O describes a particle by the wavefunction $\psi(x^\mu)$ which obeys

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x^\mu). \tag{97}$$

In another inertial frame F' with an observer O' and coordinates x'^ν given by

$$x'^\nu = \Lambda^\nu{}_\mu x^\mu, \tag{98}$$

O' describes the same particle by $\psi'(x'^\nu)$ and $\psi'(x'^\nu)$ satisfies

$$\left(i\gamma^\nu \frac{\partial}{\partial x'^\nu} - m\right)\psi'(x'^\nu). \tag{99}$$

Lorentz covariance of the Dirac equation means that the γ matrices are the same in both frames.

What is the transformation matrix S which takes ψ to ψ' under the Lorentz transformation?

$$\psi'(\Lambda x) = S\psi(x). \tag{100}$$

Applying S to Eq. (97),

$$\begin{aligned} iS\gamma^\mu S^{-1}\frac{\partial}{\partial x^\mu}S\psi(x^\mu) - mS\psi(x^\mu) &= 0 \\ \Rightarrow iS\gamma^\mu S^{-1}\frac{\partial}{\partial x^\mu}\psi'(x'^\nu) - m\psi'(x'^\nu) &= 0. \end{aligned} \tag{101}$$

Using

$$\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}, \tag{102}$$

we obtain

$$iS\gamma^\mu S^{-1}\Lambda^\nu{}_\mu \frac{\partial}{\partial x'^\nu}\psi'(x'^\nu) - m\psi'(x'^\nu) = 0. \tag{103}$$

Comparing it with Eq. (99), we need

$$S\gamma^\mu S^{-1}\Lambda^\nu{}_\mu = \gamma^\nu \quad \text{or equivalently} \quad S\gamma^\mu S^{-1} = \left(\Lambda^{-1}\right)^\mu{}_\nu \gamma^\nu. \tag{104}$$

We will write down the form of the S matrix without proof. You are encouraged to read the derivation in Shulten's notes Chapter 10, p.319-321 and verify it by yourself.

For an infinitesimal Lorentz transformation, $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon^{\mu}{}_{\nu}$. Multiplied by $g^{\nu\lambda}$ it can be written as

$$\Lambda^{\mu\lambda} = g^{\mu\lambda} + \epsilon^{\mu\lambda}, \tag{105}$$

where $\epsilon^{\mu\lambda}$ is antisymmetric in μ and λ. Then the corresponding Lorentz transformation on the spinor wavefunction is given by

$$S(\epsilon^{\mu\nu}) = I - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}, \tag{106}$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = \frac{i}{2}[\gamma_\mu,\, \gamma_\nu]. \tag{107}$$

For finite Lorentz transformation,

$$S = \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right). \tag{108}$$

Note that one can use either the active transformation (which transforms the object) or the passive transformation (which transforms the coordinates), but care should be taken to maintain consistency. We will mostly use passive transformations unless explicitly noted otherwise.

Example: Rotation about z-axis by θ angle (passive).

$$\begin{aligned}
-\epsilon^{12} &= +\epsilon^{21} = \theta, && (109)\\
\sigma_{12} &= \frac{i}{2}[\gamma_1,\, \gamma_2] = i\gamma_1\gamma_2 \\
&= i\begin{pmatrix} -i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \equiv \Sigma_3 && (110)\\
S &= \exp\left(+\frac{i}{2}\theta\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\right) = I\cos\frac{\theta}{2} + i\begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\sin\frac{\theta}{2}. && (111)
\end{aligned}$$

We can see that ψ transforms under rotations like an spin-1/2 object. For a rotation around a general direction $\hat{\boldsymbol{n}}$,

$$S = I\cos\frac{\theta}{2} + i\hat{\boldsymbol{n}}\cdot\boldsymbol{\Sigma}\sin\frac{\theta}{2}. \tag{112}$$

Example: Boost in $\hat{x}$ direction (passive).

$$\begin{aligned}
\epsilon^{01} &= -\epsilon^{10} = \eta, && (113)\\
\sigma_{01} &= \frac{i}{2}[\gamma_0,\, \gamma_1] = i\gamma_0\gamma_1, && (114)\\
S &= \exp\left(-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}\right) = \exp\left(-\frac{i}{2}\eta i\gamma_0\gamma_1\right)\\
&= \exp\left(\frac{\eta}{2}\gamma_0\gamma_1\right) = \exp\left(-\frac{\eta}{2}\alpha^1\right)\\
&= I\cosh\frac{\eta}{2} - \alpha^1\sinh\frac{\eta}{2}. && (115)
\end{aligned}$$

For a particle moving in the direction of $\hat{\boldsymbol{n}}$ in the new frame, we need to boost the frame in the $-\hat{\boldsymbol{n}}$ direction,

$$S = I\cosh\frac{\eta}{2} + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{n}}\sinh\frac{\eta}{2}. \tag{116}$$

6 Free Particle Solutions to the Dirac Equation

The solutions to the Dirac equation for a free particle at rest are

$$\begin{aligned}
\psi_1 &= \sqrt{\frac{2m}{V}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imt}, \quad E = +m,\\
\psi_2 &= \sqrt{\frac{2m}{V}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imt}, \quad E = +m,\\
\psi_3 &= \sqrt{\frac{2m}{V}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{imt}, \quad E = -m,\\
\psi_4 &= \sqrt{\frac{2m}{V}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{imt}, \quad E = -m, && (117)
\end{aligned}$$

where we have set $\hbar = c = 1$ and V is the total volume. Note that I have chosen a particular normalization

$$\int d^3x\psi^\dagger\psi = 2m \tag{118}$$

for a particle at rest. This is more convenient when we learn field theory later, because $\psi^\dagger\psi$ is not invariant under boosts. Instead, it's the zeroth component of a 4-vector, similar to E.

The solutions for a free particle moving at a constant velocity can be obtained by a Lorentz boost,

$$S = I\cosh\frac{\eta}{2} + \boldsymbol{\alpha}\cdot\hat{\boldsymbol{n}}\sinh\frac{\eta}{2}. \tag{119}$$

Using the Pauli-Dirac representation,

$$\begin{aligned}
\alpha^i &= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \\
S &= \begin{pmatrix} \cosh\frac{\eta}{2} & \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sinh\frac{\eta}{2} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sinh\frac{\eta}{2} & \cosh\frac{\eta}{2} \end{pmatrix},
\end{aligned} \tag{120}$$

and the following relations,

$$\begin{aligned}
\cosh\eta &= \gamma' = \frac{E'}{m}, \quad \sinh\eta = \gamma'\beta', \\
\cosh\frac{\eta}{2} &= \sqrt{\frac{1+\cosh\eta}{2}} = \sqrt{\frac{1+\gamma'}{2}} = \sqrt{\frac{m+E'}{2m}}, \\
\sinh\frac{\eta}{2} &= \sqrt{\frac{\cosh\eta - 1}{2}} = \sqrt{\frac{E'-m}{2m}} \\
-p'_\mu x'^\mu &= \boldsymbol{p}'_+\cdot\boldsymbol{x}' - E't' = -p_\mu x^\mu = -mt,
\end{aligned} \tag{121}$$

where $\boldsymbol{p}'_+ = |\boldsymbol{p}'_+|\hat{\boldsymbol{n}}$ is the 3-momentum of the positive energy state, we obtain

$$\begin{aligned}
\psi'_1(x') &= S\psi_1(x) = \sqrt{\frac{2m}{V}} \begin{pmatrix} \cosh\frac{\eta}{2}\begin{pmatrix}1\\0\end{pmatrix} \\ \boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\sinh\frac{\eta}{2}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} e^{-imt} \\
&= \frac{1}{\sqrt{V}}\sqrt{m+E'} \begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{\frac{E'-m}{E'+m}}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} e^{i(\boldsymbol{p}'_+\cdot\boldsymbol{x}' - E't')} \\
&= \frac{1}{\sqrt{V}}\sqrt{m+E'} \begin{pmatrix} \begin{pmatrix}1\\0\end{pmatrix} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}'_+}{E'+m}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} e^{i(\boldsymbol{p}'_+\cdot\boldsymbol{x}' - E't')}
\end{aligned} \tag{122}$$

where we have used

$$\sqrt{\frac{E'-m}{E'+m}} = \frac{\sqrt{E'^2-m^2}}{E'+m} = \frac{|\boldsymbol{p}'_+|}{E'+m} \tag{123}$$

in the last line.

$\psi_2'(x')$ has the same form except that $\begin{pmatrix}1\\0\end{pmatrix}$ is replaced by $\begin{pmatrix}0\\1\end{pmatrix}$.

For the negative energy solutions $E'_- = -E' = -\sqrt{|\boldsymbol{p}|^2 + m^2}$ and $\boldsymbol{p}'_- = v\hat{\boldsymbol{n}}E'_- = -vE'\hat{\boldsymbol{n}} = -\boldsymbol{p}'_+$. So we have

$$\psi_3'(x') = \frac{1}{\sqrt{V}}\sqrt{m+E'}\begin{pmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}'_-}{E'+m}\begin{pmatrix}1\\0\end{pmatrix} \\ \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} e^{i(\boldsymbol{p}'_-\cdot\boldsymbol{x}'+E't')}, \tag{124}$$

and $\psi_4'(x')$ is obtained by the replacement $\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$.

Now we can drop the primes and the $\pm$ subscripts,

$$\begin{aligned}\psi_{1,2} &= \frac{1}{\sqrt{V}}\sqrt{E+m}\begin{pmatrix}\chi_{+,-}\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_{+,-}\end{pmatrix} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)} = \frac{1}{\sqrt{V}}u_{1,2}e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-Et)},\\ \psi_{3,4} &= \frac{1}{\sqrt{V}}\sqrt{E+m}\begin{pmatrix}-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_{+,-}\\ \chi_{+,-}\end{pmatrix} e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+Et)} = \frac{1}{\sqrt{V}}u_{3,4}e^{i(\boldsymbol{p}\cdot\boldsymbol{x}+Et)},\end{aligned} \tag{125}$$

where

$$\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \chi_- = \begin{pmatrix}0\\1\end{pmatrix} \tag{126}$$

(V is the proper volume in the frame where the particle is at rest.)

Properties of spinors $u_1, \cdots u_4$

$$u_r^\dagger u_s = 0 \quad \text{for } r \neq s. \tag{127}$$

$$\begin{aligned} u_1^\dagger u_1 &= (E+m)\begin{pmatrix}\chi_+^\dagger & \chi_+^\dagger\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\end{pmatrix}\begin{pmatrix}\chi_+\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi_+\end{pmatrix}\\ &= (E+m)\chi_+^\dagger\left(1+\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{(E+m)^2}\right)\chi_+.\end{aligned} \tag{128}$$

Using the following identity:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = \sigma_i a_i \sigma_j b_j = (\delta_{ij} + i\epsilon_{ijk}\sigma_k)a_i b_j = \boldsymbol{a}\cdot\boldsymbol{b} + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}), \tag{129}$$

we have

$$\begin{aligned} u_1^\dagger u_1 &= (E+m)\chi_+^\dagger\left(1+\frac{|\boldsymbol{p}|^2}{(E+m)^2}\right)\chi_+\\ &= (E+m)\chi_+^\dagger\frac{E^2+2Em+m^2+|\boldsymbol{p}|^2}{(E+m)^2}\chi_+\\ &= \chi_+\frac{2E^2+2Em}{E+m}\chi_+\\ &= 2E\chi_+^\dagger\chi_+ = 2E.\end{aligned} \tag{130}$$

Similarly for other u_r we have $u_r^\dagger u_s = \delta_{rs} 2E$, which reflects that $\rho = \psi^\dagger \psi$ is the zeroth component of a 4-vector.

One can also check that

$$\overline{u}_r u_s = \pm 2m\delta_{rs} \tag{131}$$

where + for $r = 1$, 2 and − for $r = 3$, 4.

$$\begin{aligned}
\overline{u}_1 u_1 &= u_1^\dagger \gamma^0 u_1 \qquad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
&= (E+m)\chi_+^\dagger \left(1 - \frac{|\boldsymbol{p}|^2}{(E+m)^2}\right) \chi_+ \\
&= (E+m)\chi_+^\dagger \frac{E^2 + 2Em + m^2 - |\boldsymbol{p}|^2}{(E+m)^2} \chi_+ \\
&= \chi_+ \frac{2m^2 + 2Em}{E+m} \chi_+ \\
&= 2m\chi_+^\dagger \chi_+ = 2m
\end{aligned} \tag{132}$$

is invariant under Lorentz transformation.

Orbital angular momentum and spin

Orbital angular momentum

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \quad \text{or} \\
L_i &= \epsilon_{ijk} r_j p_k.
\end{aligned} \tag{133}$$

(We don't distinguish upper and lower indices when dealing with space dimensions only.)

$$\begin{aligned}
\frac{dL_i}{dt} &= i[H,\, L_i] \\
&= i[c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2,\, L_i] \\
&= ic\alpha_n [p_n,\, \epsilon_{ijk} r_j p_k] \\
&= ic\alpha_n \epsilon_{ijk} [p_n,\, r_j] p_k \\
&= ic\alpha_n \epsilon_{ijk} (-i\delta_{nj}) p_k \\
&= c\epsilon_{ijk} \alpha_j p_k \\
&= c(\boldsymbol{\alpha} \times \boldsymbol{p})_i \neq 0.
\end{aligned} \tag{134}$$

We find that the orbital angular momentum of a free particle is not a constant of the motion.

Consider the spin $\frac{1}{2}\boldsymbol{\Sigma} = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$,

$$\begin{aligned}
\frac{d\Sigma_i}{dt} &= i[H,\, \Sigma_i] \\
&= i[c\alpha_j p_j + \beta mc^2,\, \Sigma_i] \\
&= ic[\alpha_j,\, \Sigma_i]p_j \quad \left[\text{using } \Sigma_i\gamma_5 = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \alpha_i = \gamma_5\Sigma_i\right] \\
&= ic[\gamma_5\Sigma_j,\, \Sigma_i]p_j \\
&= ic\gamma_5[\Sigma_j,\, \Sigma_i]p_j \\
&= ic\gamma_5(-2i\epsilon_{ijk}\Sigma_k)p_j \\
&= 2c\gamma_5\epsilon_{ijk}\Sigma_k p_j \\
&= 2c\epsilon_{ijk}\alpha_k p_j \\
&= -2c(\boldsymbol{\alpha}\times\boldsymbol{p})_i. \qquad (135)
\end{aligned}$$

Comparing it with Eq. (134), we find

$$\frac{d(L_i + \frac{1}{2}\Sigma_i)}{dt} = 0, \qquad (136)$$

so the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \frac{1}{2}\boldsymbol{\Sigma}$ is conserved.

7 Interactions of a Relativistic Electron with an External Electromagnetic Field

We make the usual replacement in the presence of external potential:

$$\begin{aligned}
E &\rightarrow E - e\phi = i\hbar\frac{\partial}{\partial t} - e\phi, \quad e < 0 \text{ for electron} \\
\boldsymbol{p} &\rightarrow \boldsymbol{p} - \frac{e}{c}\boldsymbol{A} = -i\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}. \qquad (137)
\end{aligned}$$

In covariant form,

$$\partial_\mu \rightarrow \partial_\mu + \frac{ie}{\hbar c}A_\mu \rightarrow \partial_\mu + ieA_\mu \quad \hbar = c = 1. \qquad (138)$$

Dirac equation in external potential:

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi - m\psi = 0. \qquad (139)$$

Two component reduction of Dirac equation in Pauli-Dirac basis:

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}(E - e\phi)\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}(\boldsymbol{p} - e\boldsymbol{A})\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - m\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0,$$

$$\Rightarrow (E - e\phi)\psi_A - \boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A})\psi_B - m\psi_A = 0 \qquad (140)$$

$$-(E - e\phi)\psi_B + \boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A})\psi_A - m\psi_B = 0 \qquad (141)$$

where E and $\boldsymbol{p}$ represent the operators $i\partial_t$ and $-i\boldsymbol{\nabla}$ respectively. Define $W = E - m$, $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}$, then we have

$$\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_B = (W - e\phi)\psi_A \tag{142}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_A = (2m + W - e\phi)\psi_B \tag{143}$$

From Eq. (143),

$$\psi_B = (2m + W - e\phi)^{-1}\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_A. \tag{144}$$

Substitute it into Eq. (142),

$$\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})}{2m + W - e\phi}\psi_A = (W - e\phi)\psi_A. \tag{145}$$

In non-relativistic limit, $W - e\phi \ll m$,

$$\frac{1}{2m + W - e\phi} = \frac{1}{2m}\left(1 - \frac{W - e\phi}{2m} + \cdots\right). \tag{146}$$

In the lowest order approximation we can keep only the leading term $\frac{1}{2m}$,

$$\frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\psi_A \simeq (W - e\phi)\psi_A. \tag{147}$$

Using Eq. (129),

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\psi_A = [\boldsymbol{\pi}\cdot\boldsymbol{\pi} + i\boldsymbol{\sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{\pi})]\psi_A. \tag{148}$$

$$\begin{aligned}
(\boldsymbol{\pi}\times\boldsymbol{\pi})\psi_A &= [(\boldsymbol{p} - e\boldsymbol{A})\times(\boldsymbol{p} - e\boldsymbol{A})]\psi_A \\
&= [-e\boldsymbol{A}\times\boldsymbol{p} - e\boldsymbol{p}\times\boldsymbol{A}]\psi_A \\
&= [+ie\boldsymbol{A}\times\boldsymbol{\nabla} + ie\boldsymbol{\nabla}\times\boldsymbol{A}]\psi_A \\
&= ie\psi_A(\boldsymbol{\nabla}\times\boldsymbol{A}) \\
&= ie\boldsymbol{B}\psi_A,
\end{aligned} \tag{149}$$

so

$$\frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2\psi_A - \frac{e}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_A + e\phi\psi_A = W\psi_A. \tag{150}$$

Restoring $\hbar$, c,

$$\frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2\psi_A - \frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_A + e\phi\psi_A = W\psi_A. \tag{151}$$

This is the "Pauli-Schrödinger equation" for a particle with the spin-magnetic moment,

$$\boldsymbol{\mu} = \frac{e\hbar}{2mc}\boldsymbol{\sigma} = 2\,\frac{e}{2mc}\boldsymbol{S}. \tag{152}$$

In comparison, the relation between the angular momentum and the magnetic moment of a classical charged object is given by

$$\mu = \frac{I\pi r^2}{c} = e\frac{\omega}{2\pi}\frac{\pi r^2}{c} = \frac{e\omega r^2}{2c} = \frac{e}{2mc}m\omega r^2 = \frac{e}{2mc}L. \tag{153}$$

We can write

$$\boldsymbol{\mu} = g_s\frac{e}{2mc}\boldsymbol{S} \tag{154}$$

in general. In Dirac theory, $g_s = 2$. Experimentally,

$$g_s(e^-) = 2 \times (1.0011596521859 \pm 38 \times 10^{-13}). \tag{155}$$

The deviation from 2 is due to radiative corrections in QED, $(g-2)/2 = \frac{\alpha}{2\pi} + \cdots$. The predicted value for $g_s - 2$ using α from the quantum Hall effect is

$$(g_s - 2)_{qH}/2 = 0.0011596521564 \pm 229 \times 10^{-13}. \tag{156}$$

They agree down to the 10^{-11} level.

There are also spin-1/2 particles with anomalous magnetic moments, *e.g.,*

$$\mu_{proton} = 2.79\frac{|e|}{2m_pc}, \quad \mu_{neutron} = -1.91\frac{|e|}{2m_nc}. \tag{157}$$

This can be described by adding the Pauli moment term to the Dirac equation,

$$i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi + k\sigma_{\mu\nu}F^{\mu\nu}\psi = 0. \tag{158}$$

Recall

$$\begin{aligned}
\sigma_{\mu\nu} &= \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \\
\sigma_{0i} &= i\gamma_0\gamma_i = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -i\alpha^i, \\
\sigma_{ij} &= i\gamma_i\gamma_j = \epsilon_{ijk}\Sigma^k = \epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \\
F^{0i} &= -E^i, \\
F^{ij} &= -\epsilon^{ijk}B^k.
\end{aligned} \tag{159}$$

Then the Pauli moment term can be written as

$$i\gamma^\mu(\partial_\mu + iqA_\mu)\psi - m\psi + 2ik\boldsymbol{\alpha}\cdot\boldsymbol{E}\psi - 2k\boldsymbol{\Sigma}\cdot\boldsymbol{B}\psi = 0. \tag{160}$$

The two component reduction gives

$$(E - q\phi)\psi_A - \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_B - m\psi_A + 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E}\psi_B - 2k\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_A = 0, \tag{161}$$

$$-(E - q\phi)\psi_B + \boldsymbol{\sigma}\cdot\boldsymbol{\pi}\psi_A - m\psi_B + 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E}\psi_A - 2k\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_B = 0. \tag{162}$$

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E})\psi_B &= (W - q\phi - 2k\boldsymbol{\sigma}\cdot\boldsymbol{B})\psi_A, &(163)\\
(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E})\psi_A &= (2m + W - q\phi + 2k\boldsymbol{\sigma}\cdot\boldsymbol{B})\psi_B. &(164)
\end{aligned}
$$

Again taking the non-relativistic limit,

$$
\psi_B \simeq \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E})\psi_A, \tag{165}
$$

we obtain

$$
(W - q\phi - 2k\boldsymbol{\sigma}\cdot\boldsymbol{B})\psi_A = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} - 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E})(\boldsymbol{\sigma}\cdot\boldsymbol{\pi} + 2ik\boldsymbol{\sigma}\cdot\boldsymbol{E})\psi_A. \tag{166}
$$

Let's consider two special cases.

(a) $\phi = 0,\ \boldsymbol{E} = 0$

$$
\begin{aligned}
& (W - 2k\boldsymbol{\sigma}\cdot\boldsymbol{B})\psi_A = \frac{1}{2m}(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})^2\psi_A \\
\Rightarrow\ & W\psi_A = \frac{1}{2m}\boldsymbol{\pi}^2\psi_A - \frac{q}{2m}\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_A + 2k\boldsymbol{\sigma}\cdot\boldsymbol{B}\psi_A \\
\Rightarrow\ & \mu = \frac{q}{2m} - 2k. &(167)
\end{aligned}
$$

(b) $\boldsymbol{B} = 0,\ \boldsymbol{E} \neq 0$ for the neutron ($q = 0$)

$$
\begin{aligned}
W\psi_A &= \frac{1}{2m}\boldsymbol{\sigma}\cdot(\boldsymbol{p} + i\mu_n\boldsymbol{E})\ \boldsymbol{\sigma}\cdot(\boldsymbol{p} - i\mu_n\boldsymbol{E})\psi_A \\
&= \frac{1}{2m}\left[(\boldsymbol{p} + i\mu_n\boldsymbol{E})\cdot(\boldsymbol{p} + i\mu_n\boldsymbol{E}) + i\boldsymbol{\sigma}\cdot(\boldsymbol{p} + i\mu_n\boldsymbol{E})\times(\boldsymbol{p} - i\mu_n\boldsymbol{E})\right]\psi_A \\
&= \frac{1}{2m}\left[\boldsymbol{p}^2 + \mu_n^2\boldsymbol{E}^2 + i\mu_n\boldsymbol{E}\cdot\boldsymbol{p} - i\mu_n\boldsymbol{p}\cdot\boldsymbol{E} + i\boldsymbol{\sigma}\cdot(i\mu_n\boldsymbol{p}\times\boldsymbol{E} - i\mu_n\boldsymbol{E}\times\boldsymbol{p})\right]\psi_A \\
&= \frac{1}{2m}\left[\boldsymbol{p}^2 + \mu_n^2\boldsymbol{E}^2 - \mu_n(\boldsymbol{\nabla}\cdot\boldsymbol{E}) + 2\mu_n\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p}) + i\mu_n\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E})\right]\psi_A \\
&= \frac{1}{2m}\left[\boldsymbol{p}^2 + \mu_n^2\boldsymbol{E}^2 - \mu_n\rho + 2\mu_n\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p})\right]\psi_A. &(168)
\end{aligned}
$$

The last term is the spin-orbit interaction,

$$
\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{p}) = -\frac{1}{r}\frac{d\phi}{dr}\boldsymbol{\sigma}\cdot(\boldsymbol{r}\times\boldsymbol{p}) = -\frac{1}{r}\frac{d\phi}{dr}\boldsymbol{\sigma}\cdot\boldsymbol{L}. \tag{169}
$$

The second to last term gives an effective potential for a slow neutron moving in the electric field of an electron,

$$
V = -\frac{\mu_n\rho}{2m} = \frac{\mu_n}{2m}(-e)\delta^3(\boldsymbol{r}). \tag{170}
$$

It's called "Foldy" potential and does exist experimentally.