PHY 110 C
HW #4

4a $V = \frac{-p_0\omega}{4\pi\varepsilon_0 c}\left[\frac{x}{r^2}\sin\omega(t-r/c) - \frac{y}{r^2}\cos\omega(t-r/c)\right]$

(using cartesians, and noting $\sin(\theta-\pi/2) = -\cos\theta$)

$$V = (r,\theta,\phi,t) = \frac{-p_0\omega}{4\pi\varepsilon_0 c}\frac{\sin\theta}{r}\left[\cos\phi\sin\omega(t-r/c) - \sin\phi\cos\omega(t-r/c)\right]$$

likewise:

$$\vec{A}(x,y,z,t) = \frac{-\mu_0 p_0\omega}{4\pi r}\left[\sin\omega(t-r/c)\,\hat{x} - \cos\omega(t-r/c)\,\hat{y}\right]$$

$$\vec{E} = \frac{\mu_0 p_0\omega^2}{4\pi r}\cos\omega(t-r/c)\left[\hat{z} - \frac{z}{r}\hat{r}\right]$$

$$\Rightarrow \boxed{\vec{E} = \frac{\mu_0 p_0\omega^2}{4\pi r}\left\{\cos\omega(t-r/c)\left[\hat{x} - \frac{x}{r}\hat{r}\right] + \sin\omega(t-r/c)\left[\hat{y} - \frac{y}{r}\hat{r}\right]\right\}}$$

$$\boxed{\vec{B} = \frac{1}{c}\left(\hat{r}\times\vec{E}\right)}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{E^2}{\mu_0 c}\hat{r}$$

$$E^2 = \left[\frac{\mu_0 p_0\omega^2}{4\pi r}\right]^2\left[1 - \left(\sin\theta\cos\{w(t-r/c)+\phi\}\right)^2\right] \quad \text{note: } \cos\phi\cos\omega t_r - \sin\phi\sin\omega t_r = \cos(\omega t_r + \phi)$$

$$\Rightarrow \boxed{\vec{S} = \frac{\mu_0}{c}\left(\frac{p_0\omega}{4\pi r}\right)^2\left[1 - \left(\sin\theta\cos\{\omega(t-r/c)+\phi\}\right)^2\right]\hat{r}}$$

$$\boxed{\langle\vec{S}\rangle = \frac{\mu_0}{c}\left(\frac{p_0\omega}{4\pi r}\right)^2\left[1 - \frac{1}{2}\sin^2\theta\right]\hat{r}}$$

$$P = \int\langle\vec{S}\rangle\cdot d\vec{a} = \frac{\mu_0 p_0{}^2\omega^4}{8\pi c}\left[2 - \frac{1}{2}\frac{4}{3}\right] = \boxed{\frac{\mu_0 p_0{}^2\omega^4}{6\pi c}}$$

4b $\vec{\mathcal{J}}(\vec{r}) = -i\,k\,m_0 \sin\theta\,\hat{\phi}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\vec{\mathcal{J}}(\vec{r}) = \frac{-\mu_0}{4\pi}\,i\,k\,m_0\,\frac{\sin\theta}{r}\,e^{ikr}\,\phi$$

$$\therefore \quad \boxed{\vec{A}(\vec{r},t) = \frac{-\mu_0}{4\pi}\,i\,k\,m_0\,\frac{\sin\theta}{r}\,e^{i(kr-\omega t)}\,\hat{\phi}}$$

$$\vec{E}(\vec{r},t) = \frac{-\partial\vec{A}}{\partial t} = \frac{\mu_0}{4\pi}\frac{\omega}{c}\,m_0\,\frac{\sin\theta}{r}\,\omega\,e^{i(kr-\omega t)} \qquad k = \frac{\omega}{c}; \quad \text{no } V$$

Take the real part:

$$\boxed{\vec{E}(r,t) = \frac{\mu_0}{4\pi}\frac{\omega^2}{c}\,m_0\,\frac{\sin\theta}{r}\cos\omega(t-r/c)\,\hat{\phi}} \qquad \cos(-\theta) = \cos\theta$$

$$\nabla\times\vec{A} = \frac{\mu_0}{4\pi}\frac{i\omega}{c}\,m_0\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin^2\theta}{r}\,e^{i(kr-\omega t)}\,\hat{r} - \frac{1}{r}\frac{\partial r}{\partial r}\frac{\sin\theta}{r}\,e^{i(kr-\omega t)}\,\hat{\theta}\right\}$$

$$= \frac{\mu_0}{4\pi}\frac{i\omega}{c}\,m_0\left\{\frac{2\cos\theta}{r^2}\,e^{i(kr-\omega t)}\,\hat{r} - \frac{\sin\theta}{r}\,i\,k\,e^{i(kr-\omega t)}\,\hat{\theta}\right\}$$

$$\boxed{\vec{B} = \frac{-\mu_0}{4\pi}\frac{\omega}{c}\,m_0\,\frac{2\cos\theta}{r^2}\sin\omega(t-r/c)\,\hat{r} - \frac{\mu_0}{4\pi}\frac{\omega^2}{c^2}\,m_0\,\frac{\sin\theta}{r}\cos\omega(t-r/c)\,\hat{\theta}}$$

Let $\quad E_0 \equiv \frac{\mu_0}{4\pi}\frac{\omega^2}{c}\,m_0 \qquad u \equiv \omega(r/c-t) \qquad \omega(t-r/c) = -u$

$$\vec{E} = E_0\,\frac{\sin\theta}{r}\cos u\,\phi$$

$$\vec{B} = \frac{E_0}{\omega}\frac{2\cos\theta}{r^2}\sin u\,\hat{r} - \frac{E_0}{c}\cdot\frac{\sin\theta}{r}\cos u\,\hat{\theta} \qquad \sin(-u) = -\sin u$$

$$\vec{S} = \frac{E_0{}^2}{\mu}\left\{\frac{2\cos\theta\sin\theta}{w\,r^3}\cos u\sin u\,\hat{\theta} + \frac{1}{c}\frac{\sin^2\theta}{r^2}\cos^2 u\,\hat{r}\right\}$$

$$\boxed{\vec{S} = \frac{\mu_0}{16\pi^2}\frac{\omega^3}{c^2}\left\{\frac{-2\cos\theta\sin\theta}{r^3}\cos\omega(t-r/c)\sin\omega(t-r/c)\,\hat{\theta} + \frac{\omega}{c}\frac{\sin^2\theta}{r^2}\cos^2\omega(t-r/c)\,\hat{r}\right\}}$$

$$\boxed{\langle\vec{S}\rangle = \frac{\mu_0}{32\pi^2}\frac{\omega^4}{c^4}\,m_0{}^2\,\frac{\sin^2\theta}{r^2}\,\hat{r}}$$

4c $\vec{p}(t) = p \cdot (\cos\omega t\, x + \sin\omega t\, y) \Rightarrow \ddot{\vec{p}}(t) = -\omega^2 p_0 (\cos\omega t\, x + \sin\omega t\, y)$

$$\therefore \quad \left(\ddot{\vec{p}}(t)\right)^2 = \omega^4 p_0{}^2 \left((\cos\omega t)^2 + (\sin\omega t)^2\right) = p_0{}^2\omega^4$$

11.59 says: $\boxed{\vec{S} = \frac{\mu_0 p_0{}^2 \omega^4}{16\pi^2 c} \frac{\sin^2\theta}{r^2} \hat{r}}$

(This appears to disagree with the answer to problem 11.4. The reason is that in 11.59 the polar axis is along the direction of $\ddot{\vec{p}}(t_0)$ - as the dipole rotates, so do the axes. So the angle θ here is not the same as in problem 11.4.)

11.60 says: $\boxed{P = \frac{\mu_0 p_0{}^2 \omega^4}{6\pi c}}$

(P does agree with problem 11.4, for now we have integrated over all angles, and the orientation of the polar axis is irrelevant.)

4d $\vec{p}_0 = \int \lambda \vec{r}\, d\ell = \pi R^2 \lambda_0 \hat{y}$

$$\vec{p}(t) = p_0 (\cos\omega t\, \hat{x} - \sin\omega t\, \hat{y}) \qquad \ddot{\vec{p}} = -\omega^2 \vec{p}$$

$$(11.56) \quad \Rightarrow \quad \vec{E} = \frac{\mu_0}{4\pi r} \hat{r} \times (\hat{r} \times \ddot{\vec{p}}) = -c(\hat{r} \times \vec{B})$$

$$\Rightarrow \quad \vec{S} = \frac{c}{\mu_0} B^2\, \hat{r} = \frac{\mu_0}{16\pi^2 c r^2} (\hat{r} \times \ddot{\vec{p}})^2\, \hat{r}$$

$$\Rightarrow \quad P = \int \vec{S} \cdot d\vec{a} = \frac{\mu_0}{16\pi^2 c r^2} \int \frac{(\hat{r} \times \ddot{\vec{p}})^2}{r^2}\, \hat{r} \cdot \hat{r}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 \omega^4 p_0{}^2}{16\pi^2 c} \int (\hat{r} \times \hat{p})^2 \sin\theta\, d\theta$$

$$\hat{r} \times \hat{p} = \hat{x}(-\cos\theta \sin\omega t) - \hat{y}(\cos\theta \cos\omega t) + \hat{z}\, [\sin\theta \cos\phi(-\sin\omega t) - \sin\theta \sin\phi \cos\omega t]$$

$$(\hat{r} \times \hat{p})^2 = \cos^2\theta + \sin^2\theta \sin^2(\phi - \omega t)$$

$$\int (\hat{r} \times \hat{p})^2 \sin\theta\, d\theta\, d\phi = \int_0^\pi \cos^2\theta \sin\theta\, d\theta \int_0^{2\pi} d\phi + \int_0^\pi \sin^2\theta \sin\theta\, d\theta \int_0^{2\pi} \sin^2(\phi - \omega t) d\phi$$

$$= \frac{2}{3} \cdot 2\pi + \frac{4}{3} \cdot \pi = \frac{8}{3}\pi$$

$$\boxed{P = \frac{\mu_0 \omega^4 p_0{}^2}{c 6\pi}}$$