Physics 115A Spring 2006

Problem Set 5
Due by 4:00 pm Wednesday, 5/24 (in class, at my office, or in my mailbox)

1. **S-matrix** (15 points)
Consider the delta function potential well

$$V(x) = -\alpha\delta(x)$$

with $\alpha > 0$.
  a. Find the scattering matrix S for this potential.
  b. Show that S is unitary, that is, $S^\dagger S = 1$.
  c. Show that the S-matrix blows up at the complex value $k = i\kappa$, where κ has exactly the value determined by the bound state energy.

(Note: you may use the results of Griffiths section 2.5—you don't need to recalculate the bound state energy or the matching conditions for the coefficients of the wave function.)

2. **Hermitian operators** (10 points)
  a. In momentum space—that is, the space of wave functions $\phi(p)$ of momentum—the position operator is $\hat{x} = -\frac{\hbar}{i}\frac{d}{dp}$. Show that $\hat{x}$ is hermitian.
  b. For a particle in two dimensions described in polar coordinates, the inner product is

$$\langle\chi|\psi\rangle = \int \chi^*(r,\theta)\psi(r,\theta)rdrd\theta$$

Show that the operator $\frac{\hbar}{i}\frac{d}{dr}$ is *not* hermitian, and find the constant c for which the operator $p_r = \frac{\hbar}{i}\frac{d}{dr} + \frac{c}{r}$ *is* hermitian.

3. **Miscellaneous algebra** (10 points)
  a. Prove that $(AB)^\dagger = B^\dagger A^\dagger$.
  b. Find the eigenvalues and eigenfunctions of the matrix

$$M = \begin{pmatrix} \alpha & \epsilon \\ -\epsilon & \alpha \end{pmatrix}$$

4. **Some operator algebra** (15 points)
The exponential of an operator is defined by a power series,

$$e^{\hat{O}} = 1 + \hat{O} + \frac{1}{2!}\hat{O}^2 + \frac{1}{3!}\hat{O}^3 + \dots$$

a. Suppose $\hat{A}$ is hermitian. Show that

$$\left(e^{i\hat{A}}\right)^\dagger = e^{-i\hat{A}}$$

(Note that this implies that if $\hat{A}$ is hermitian, then $e^{i\hat{A}}$ is unitary.)

b. Let $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$ be the usual momentum operator. Define $U(a) = e^{iax/\hbar}$ and $V(b) = e^{ib\hat{p}/\hbar}$.

i. Find $U^\dagger(a)\hat{p}U(a)$.

ii. Find $V^\dagger(b)xV(b)$.

Hint: there's a nice trick for the last question. Define $f(b) = V^\dagger(b)xV(b)$, and work out $df(b)/db$. This will give you a differential equation for $f(b)$ that should be easy to solve.