Some Useful Facts for Physics 115A

Hilbert spaces: states $|\psi\rangle$; inner product $\langle\chi|\psi\rangle = \langle\psi|\chi\rangle^*$; $\langle\chi|c\psi\rangle = c\langle\chi|\psi\rangle$ and $\langle c\chi|\psi\rangle = c^*\langle\chi|\psi\rangle$
orthonormality: $\langle e_i|e_j\rangle = \delta_{ij}$; completeness: $|\psi\rangle = \sum\langle e_i|\psi\rangle|e_i\rangle$ for any $|\psi\rangle$
for continuous states, orthonormality: $\langle\alpha|\beta\rangle = \delta(\alpha-\beta)$; completeness: $|\psi\rangle = \int d\alpha\langle\alpha|\psi\rangle|\alpha\rangle$
Schwarz inequality $|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$

Operators: adjoint $\langle\chi|\hat{A}\psi\rangle = \langle\hat{A}^\dagger\chi|\psi\rangle$; hermitian $\Rightarrow$ $\hat{A}^\dagger = \hat{A}$; unitary $\Rightarrow$ $\hat{U}^\dagger\hat{U} = 1$
matrix elements $A_{ij} = \langle e_i|\hat{A}|e_j\rangle$; $\langle e_i|\hat{A}^\dagger|e_j\rangle = \langle e_j|\hat{A}|e_i\rangle^*$

Commutators: $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, $[\hat{x},\hat{p}] = i\hbar$

Eigenstates and eigenvalues: $\hat{A}|a\rangle = a|a\rangle$; $\hat{A}$ hermitian $\Rightarrow a$ real
if eigenstate $|a\rangle$ can be written as $|a\rangle = \sum_i c_i|e_i\rangle$ then $\sum_j A_{ij}c_j = ac_i$
characteristic equation: $\det|A_{ij} - a\delta_{ij}| = 0$
(generalized) eigenstates of a hermitian operator are (usually) complete

Momentum operator: $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Hamiltonian: $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$

Position and momentum representations: $|a\rangle = \delta(x-a)$, $|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$
$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int dx\, e^{-ipx/\hbar}\psi(x)$

General probabilities: given state $|\psi\rangle$, observable $\hat{A}$ with eigenstate $|a\rangle$:
$\text{Prob}(A=a) = |\langle\psi|a\rangle|^2$

Probabilities: position and momentum $\text{Prob}(a < x < b \text{ at time } t) = \int_a^b \psi^*(x,t)\psi(x,t)dx$;
$\text{Prob}(a < p < b) \text{ at time } t) = \int_a^b \phi^*(x,t)\phi(x,t)dx$

Energy eigenstates: $\int\psi_m^*(x)\psi_n(x)dx = \delta_{mn}$, $\psi(x) = \sum_n c_n\psi_n(x)$, with $c_n = \int\psi_n^*(x)\psi(x)dx$
and $\text{Prob}(E = E_n) = |c_n|^2$

Probability current: $\frac{\partial}{\partial t}(\psi^*\psi) = -\frac{\partial}{\partial x}J$, with $J(x) = \frac{\hbar}{2im}\left(\psi^*\frac{\partial\psi}{\partial x} - \psi\frac{\partial\psi^*}{\partial x}\right)$

Expectation values: $\langle\hat{\mathcal{O}}(x,p)\rangle = \langle\psi|\hat{\mathcal{O}}|\psi\rangle = \int_{-\infty}^{\infty}\psi^*(x,t)\hat{\mathcal{O}}(x,\frac{\hbar}{i}\frac{\partial}{\partial x})\psi(x,t)dx$

Uncertainties: $\Delta\hat{A} = \left(\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2\right)^{1/2}$

General uncertainty principle: in state $|\psi\rangle$, $\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|$

Energy-time uncertainty principle: $\Delta E\Delta t \geq \frac{\hbar}{2}$ with $\Delta t = \Delta\hat{B}/(d\langle\hat{B}\rangle/dt)$

Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \hat{H}\psi(\mathbf{x},t)$

Time-independent Schrödinger equation: $\hat{H}\psi(\mathbf{x}) = E\psi(\mathbf{x})$; $\psi(\mathbf{x},t) = e^{-iEt/\hbar}\psi(\mathbf{x},0)$

Evolution of expectation values: $\frac{d\langle\hat{A}\rangle}{dt} = \langle\frac{\partial\hat{A}}{\partial t}\rangle + \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$

Continuity: $\psi(x)$ is continuous everywhere; $\frac{d\psi}{dx}$ is continuous except at points where V is infinite;
for delta function potential, integral of Schrödinger equation gives additional matching condition

Group and phase velocity: $E = \hbar\omega$, $p = \hbar k$
Given $\omega = \omega(k)$ (dispersion relation), $v_{\text{group}} = \frac{d\omega}{dk}$ $v_{\text{phase}} = \frac{\omega}{k}$

Free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$

Plane wave: $\Psi(x,t) = Ae^{ikx-i\omega t}$, $\quad k = \frac{p}{\hbar} = \frac{\sqrt{2mE}}{\hbar}$, $\quad \omega = \frac{E}{\hbar}$

Superposition: $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int \Phi(p,t) e^{ipx/\hbar} dp$

Infinite square well: $\hat{H} = \frac{\hat{p}^2}{2m} + V, \quad V = \begin{cases} 0 & 0 \le x \le a \\ \infty & elsewhere \end{cases}$

$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \quad E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

Harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \quad E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}} (\mp i\hat{p} + m\omega x), \quad \hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}_+ + \hat{a}_-), \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} (\hat{a}_+ - \hat{a}_-)$

$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$

$[\hat{a}_-, \hat{a}_+] = 1$

$\hat{a}_-\psi_0 = 0, \quad \hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$\psi_n(x) = c_n H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right) e^{-\frac{m\omega}{2\hbar}x^2}$

Piecewise constant potentials: suppose $V = V_0$ is constant in some region

If $E > V_0$: $\frac{2m(E-V_0)}{\hbar^2} = k^2$, $\psi(x) = A\sin kx + B\cos kx = \tilde{A}e^{ikx} + \tilde{B}e^{-ikx}$

If $E < V_0$: $\frac{2m(E-V_0)}{\hbar^2} = -\kappa^2$, $\psi(x) = Ce^{\kappa x} + De^{-\kappa x} = \tilde{C}\cosh\kappa x + \tilde{D}\sinh\kappa x$

$e^{ikx} \Leftrightarrow$ right-moving, $\quad e^{-ikx} \Leftrightarrow$ left-moving

Reflection and transmission: for incoming free particle from left ($x \ll 0$):

$\psi(x) = e^{ikx} + re^{-ikx}$ for $x \ll 0$; $\psi(x) = te^{ikx}$ for $x \gg 0$

reflection probability $R = |r|^2$, transmission probability $T = |t|^2$

Scattering matrix: for particle that is free for $x \ll 0$ and $x \gg 0$:

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ for $x \ll 0$; $\psi(x) = Fe^{ikx} + Ge^{-ikx}$ for $x \gg 0$

$\begin{pmatrix} B \\ F \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ G \end{pmatrix} = \mathbf{S} \begin{pmatrix} A \\ G \end{pmatrix}$, with $\mathbf{S}$ unitary