Some Useful Facts for Physics 115A

Probabilities: $\text{Prob}(a < x < b \text{ at time } t) = \int_a^b \Psi^*(x,t)\Psi(x,t)dx$

Probability current: $\frac{\partial}{\partial t}(\Psi^*\Psi) = -\frac{\partial}{\partial x}J$, with $J(x) = \frac{\hbar}{2im}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \Psi\frac{\partial\Psi^*}{\partial x}\right)$

Expectation values: $\langle\hat{\mathcal{O}}(x,p)\rangle = \int_{-\infty}^{\infty}\Psi^*(x,t)\hat{\mathcal{O}}(x,\frac{\hbar}{i}\frac{\partial}{\partial x})\Psi(x,t)dx$

Uncertainties: $\Delta x = \left(\langle x^2\rangle - \langle x\rangle^2\right)^{1/2}$, $\Delta p = \left(\langle p^2\rangle - \langle p\rangle^2\right)^{1/2}$

Schrödinger Equation: $i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \hat{H}\Psi(\mathbf{x},t)$

Hamiltonian: $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\nabla^2 + V(x)$

Time-independent Schrödinger Equation: $\hat{H}\psi(\mathbf{x}) = E\psi(\mathbf{x})\,;\quad \psi(\mathbf{x},t) = e^{-iEt/\hbar}\psi(\mathbf{x},0)$

Momentum operator: $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

Orthonormality and completeness: For energy eigenstates,

$\int\psi_m^*(x)\psi_n(x)dx = \delta_{mn}$, $\psi(x) = \sum_n c_n\psi_n(x)$, with $c_n = \int\psi_n^*(x)\psi(x)dx$

and $\text{Prob}(E = E_n) = |c_n|^2$

Solving the Time-Dependent Schrödinger Equation:

Given $\Psi(x,0)$, write $\Psi(x,0) = \sum_n c_n\psi_n(x)$ (where ψ_n are energy eigenstates)

Then $\Psi(x,t) = \sum_n c_n\psi_n(x)e^{-iE_nt/\hbar}$

Free particle: $\hat{H} = \frac{\hat{p}^2}{2m}$

Plane wave: $\psi(x) = Ae^{ikx-i\omega t}$, $k = \frac{p}{\hbar} = \frac{\sqrt{2mE}}{\hbar}$, $\omega = \frac{E}{\hbar}$

Superposition: $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int\Phi(p,t)e^{ipx/\hbar}dp$

with $\text{Prob}(a \le p \le b \text{ at time } t) = \int_a^b\Phi^*(p,t)\Phi(p,t)dp$

Group velocity: $v_{group} = d\omega/dk = dE/dp$

Infinite square well: $\hat{H} = \frac{\hat{p}^2}{2m} + V$, $V = \begin{cases} 0 & 0 \le x \le a \\ \infty & elsewhere \end{cases}$

$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$

Harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$

$\hat{a}_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(\mp i\hat{p} + m\omega x)$, $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$, $\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}_+ - \hat{a}_-)$

$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$

$[\hat{a}_-, \hat{a}_+] = 1$

$\hat{a}_-\psi_0 = 0$, $\hat{a}_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$, $\hat{a}_-\psi_n = \sqrt{n}\,\psi_{n-1}$

$\psi_n(x) = a_nH_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2}$