Probability Current

Let R be some region in space.
(Notation: the boundary of R—that is, the surface surrounding R—is often denoted ∂R.)

Suppose a particle has a wave function $\psi(\mathbf{x}, t)$. We are interested in the probability of finding the particle in the region R:

$$\text{Prob}(R) = \int_R |\psi|^2 d^3x$$

Since there is some chance of the particle moving into or out of R, this probability can change in time. Its time derivative is (since $|\psi|^2 = \psi^*\psi$)

$$\frac{d}{dt}\text{Prob}(R) = \int_R \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t} \right) d^3x \tag{1}$$

Now use the Schrödinger equation and its complex conjugate:

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \\ -i\hbar \frac{\partial \psi^*}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* \end{aligned} \tag{2}$$

Solve for $\frac{\partial \psi}{\partial t}$ and $\frac{\partial \psi^*}{\partial t}$, and insert into (1) to obtain

$$\frac{d}{dt}\text{Prob}(R) = -\int_R \frac{i\hbar}{2m}\left(\psi\nabla^2\psi^* - \psi^*\nabla^2\psi\right) d^3x = -\int_R \nabla \cdot \mathbf{J} d^3x \tag{3}$$

where

$$\mathbf{J} = \frac{i\hbar}{2m}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) \tag{4}$$

(Check the last equality in equation (3)—do the derivatives!) Then by Stokes theorem (or Green's theorem), equation (3) becomes

$$\frac{d}{dt}\text{Prob}(R) = -\int_{\partial R} \mathbf{J} \cdot \mathbf{n} d^2x \tag{5}$$

(Depending on the place you learned this, you may have seen the last term written instead as $-\int \mathbf{J} \cdot d\mathbf{A}$ or $-\int \mathbf{J} \cdot d\mathbf{S}$.)

Equation (5) should look familiar from E&M: it's the same form as the equation for charge conservation. It says that the probability in R can change only by an amount equal to the flux of the "probability current" $\mathbf{J}$ through the surface surrounding R. The current $\mathbf{J}$ therefore describes the flow of probability, the nearest thing we have in quantum mechanics to a description of the motion of a particle. We will be using this *a lot* in this course.

(See Griffiths problem 1.14 for the one-dimensional version.)