Raising and lowering operators: a slightly simpler derivation

Suppose we have an energy eigenstate for the harmonic oscillator,

$$\hat{H}\psi = E\psi$$

Consider the new wave function $\tilde{\psi} = \hat{a}_+\psi$. We have

$$\begin{aligned}
\hat{H}\tilde{\psi} &= \hat{H}\hat{a}_+\psi \\
&= \left(\hat{H}\hat{a}_+ - \hat{a}_+\hat{H} + \hat{a}_+\hat{H}\right)\psi \qquad \text{(adding zero)} \\
&= \left([\hat{H},\hat{a}_+] + \hat{a}_+\hat{H}\right)\psi \\
&= [\hat{H},\hat{a}_+]\psi + \hat{a}_+E\psi \qquad \text{(since } \hat{H}\psi = E\psi\text{)} \\
&= [\hat{H},\hat{a}_+]\psi + E\hat{a}_+\psi = [\hat{H},\hat{a}_+]\psi + E\tilde{\psi} \qquad (1)
\end{aligned}$$

So we need the commutator $[\hat{H},\hat{a}_+]$. Now,

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

so

$$\begin{aligned}
[\hat{H},\hat{a}_+] &= \hbar\omega\left\{\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\hat{a}_+ - \hat{a}_+\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\right\} \\
&= \hbar\omega\left\{\hat{a}_+\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_+\hat{a}_-\right\} \\
&= \hbar\omega\left\{\hat{a}_+[\hat{a}_-,\hat{a}_+]\right\} \\
&= \hbar\omega\hat{a}_+ \qquad \text{(since } [\hat{a}_-,\hat{a}_+] = 1\text{)}
\end{aligned}$$

Thus from equation (1),

$$\hat{H}\tilde{\psi} = [\hat{H},\hat{a}_+]\psi + E\tilde{\psi} = \hbar\omega\hat{a}_+\psi + E\tilde{\psi} = (E + \hbar\omega)\tilde{\psi}$$

Similarly, for the wave function $\check{\psi} = \hat{a}_-\psi$,

$$\hat{H}\check{\psi} = [\hat{H},\hat{a}_-]\psi + E\check{\psi}$$

and

$$[\hat{H},\hat{a}_-] = \hbar\omega\left\{\hat{a}_+\hat{a}_-\hat{a}_- - \hat{a}_-\hat{a}_+\hat{a}_-\right\} = \hbar\omega\left\{[\hat{a}_+,\hat{a}_-]\hat{a}_-\right\} = -\hbar\omega\hat{a}_-$$

since $[\hat{a}_+,\hat{a}_-] = -[\hat{a}_-,\hat{a}_+] = -1$. Thus

$$\hat{H}\check{\psi} = -\hbar\omega\hat{a}_-\psi + E\check{\psi} = (E - \hbar\omega)\check{\psi}$$