$\boxed{\text{Laplace transform "appendix"}}$

Active filters & how they work

Ref: Text, various places; Abramowitz & Stegun, Hbk. of Math. Funcs., Ch. 29
Millman, Microelectronics, Sec. 16-6
Kowalski, Nuclear Electronics, Appendix 8

Filters are conveniently analyzed using the Laplace transform techniques mentioned earlier. One deals with the circuit in the Laplace transformed complex frequency space using circuit analysis which is closely analogous to what was done before using complex impedance. In particular, we get the following:

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

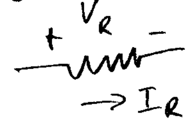
where $\hat{f}(s)$ is the transformed quantity.

$$\boxed{s = \sigma + j\omega}$$

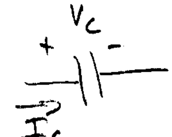
Furthermore

$\hat{f}(s)$	$f(t)$	
c	$c\,\delta(t)$	
c/s	$c\,\theta(0)$	← const. times unit step fcn. (written here explicitly)
$s\hat{f}(s) - f(+0)$	$\frac{df}{dt}$	(+0 means approached from above)
$\frac{1}{s}\hat{f}(s)$	$\int_0^t f(\tau)\,d\tau$	
$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$	poles of $\hat{f}(s)$ in s-space determine the behavior of $f(t)$.
$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	The δ-function response looks like $f(t)$ up to a constant and a $\theta(0)$.
$\frac{1}{s+a}$	e^{-at}	More poles? Use superposition.

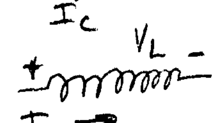
$\hat{V}_R(s) = R\hat{I}_R(s)$ $\qquad V_R(t) = R I_R(t)$



* $\hat{V}_C(s) = \frac{1}{sC}\hat{I}_C(s)$ $\qquad V_C(t) = \frac{1}{C}\int_0^t I_C(\tau)\,d\tau$



** $\hat{V}_L(s) = sL\hat{I}_L(s)$ $\qquad V_L(t) = L\frac{d}{dt}I_L(t)$



(note i direction)

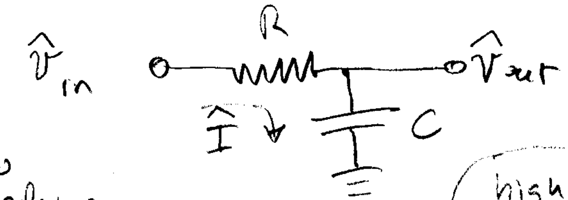
*,** Note: These require $\begin{cases} V_C = 0 \text{ when } t=0 \ (*) \\ I_L = 0 \quad " \quad " \ (**) \end{cases}$

(Other initial conditions can also be done)

$te^{-at}u(t) = \frac{1}{(s+a)^2}$

Examples -

We saw for the low pass filter



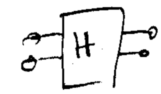
low impedance source

high impedance load

transfer fcn:

$$\underset{\sim}{H}(j\omega) \equiv \frac{\tilde{V}_{out}}{\tilde{V}_{in}}$$

$$\hat{H}(s) \equiv \frac{\hat{V}_{out}}{\hat{V}_{in}}$$



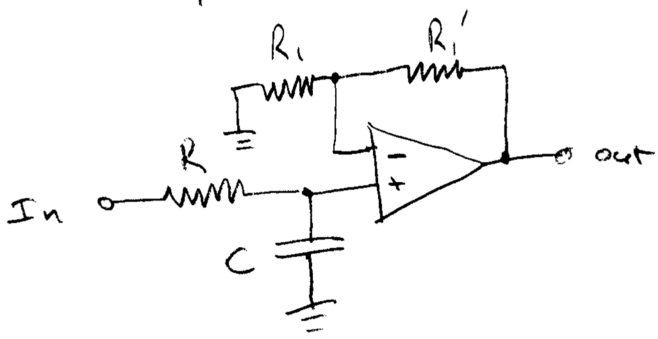
$$\left.\begin{aligned} \hat{V}_{out} &= \frac{1}{sC}\hat{I} \\ \hat{V}_{in} &= \hat{I}R + \frac{1}{sC}\hat{I} \end{aligned}\right\} \Rightarrow \hat{H}(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{\omega_c}{s + \omega_c}$$

δ-function response is exponential and

$$|H(\omega)| = \frac{1}{\sqrt{1+(\omega/\omega_c)^2}} \quad \text{where } \omega_c = \frac{1}{RC} \quad (-3\text{dB point}).$$

(above ω_B A_v falls at 20dB/decade)

→ Now we want to improve the filter response - we could "activize" the filter and cascade many stages to get more dB/decade;



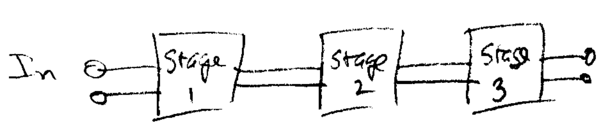
$\omega_B = \frac{1}{RC}$, $A_{V_0} = 1 + \frac{R_1'}{R_1}$ (arbitrary)

A stage

$$\left|\frac{A_v}{A_{v_0}}\right| = \frac{1}{\sqrt{1+(\omega/\omega_c)^2}}$$

where A_{v_0} = "passband transfer fcn" (at $\omega = 0$ here)

So



gives 60 dB/decade 3-pole RC filter.

(ok up to a few MHz with suitable op amps)

Improve things further by making RLC filters (or "activized" filters with just R's & C's)

Study has shown that certain arrangements of poles are advantageous for different situations – e.g. the Butterworth filter has a sharper cutoff near ω_B than a cascaded RC filter with the same number of poles while retaining a smooth A_V in the "pass band."

$$\frac{A_V(s)}{A_{V_0}} = \frac{1}{B_n(s)} \qquad \left|\frac{A_V(\omega)}{A_{V_0}}\right| = \frac{1}{|B_n(\omega)|} = \frac{1}{\sqrt{1+(\omega/\omega_B)^{2n}}}$$

magnitude

where

n	$B_n(s)$ (Butterworth polynomials)
1	$(s/\omega_B)+1$ ← This one is our RC filter
2	$(s/\omega_B)^2 + 1.414(s/\omega_B)+1$
3	$(s/\omega_B+1)((s/\omega_B)^2 + s/\omega_B + 1)$
4	$((s/\omega_B)^2 + 0.765\, s/\omega_B + 1)((s/\omega_B)^2 + 1.848\, s/\omega_B + 1)$

etc.

The actual filters can be made in sections with the first order terms realized by an RC circuit and the 2nd order terms by an LRC circuit or an op amp circuit with R's and C's only; e.g., for a 3-pole filter:

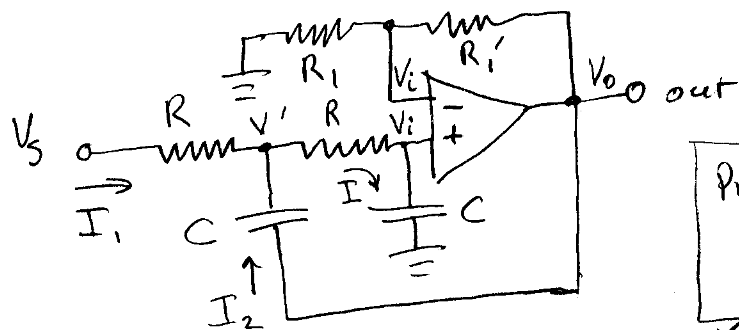
first-order term as before with RC case

second order term with 2 C's & opamp (+ R's) (see next page)

The 2nd order polynomial term looks in general like:

$$\frac{A_V(s)}{A_{V_0}} = \frac{1}{(s/\omega_B)^2 + 2k(s/\omega_B) + 1}$$

It can be realized using



$A_V(s) = V_o/V_s$

Problem: Find $\dfrac{A_V(s)}{A_{V_0}}$

Analyze the circuit using Kirchhoff's laws as before — work in s-space. I will not use the "hats" because I have to write V and I so much — You will see we are in s-space because things are fcns of s.

For the op amp, $V_+ = V_- = V_i$. No current flows into ⊕ or ⊖.

A_{V_0} = midband gain = $1 + R_1'/R_1$ (a constant)

Now $I = V_i / \frac{1}{sC} = sCV_i$

$\underline{\underline{V'}} = I(R + \frac{1}{sC}) = sCV_i(R + \frac{1}{sC}) = \boxed{\frac{V_o}{A_{V_0}}(sRC+1)}$

$$\Rightarrow I = \frac{V_o}{A_{V_0}} \frac{sRC+1}{R + 1/sC} = \frac{V_o}{A_{V_0}} sC$$

$$I_2 = (V_o - V')sC = I - I_1 \Rightarrow I_1 = I - (V_o - V')sC$$

$$V_S = V' + I_1 R = V' + IR - (V_o - V')sRC$$
$$= \frac{V_o}{A_{V_0}}(sRC+1) + \frac{V_o}{A_{V_0}} sRC - V_o sRC + \frac{V_o}{A_{V_0}}(sRC+1)sRC$$

Thus $\dfrac{A_{V_0}}{(V_o/V_s)} = sRC + 1 + sRC - A_{V_0} sRC + (sRC)^2 + sRC = \underline{(sRC)^2 + (3 - A_{V_0})sRC + 1}$

or $\boxed{\dfrac{A_V(s)}{A_{V_0}} = \dfrac{1}{(sRC)^2 + (3 - A_{V_0})sRC + 1}}$ $\left\{ \begin{array}{l} \text{choose } RC = \frac{1}{\omega_B} \\ \text{and } A_{V_0} = 3 - 2k = 1 + R_1'/R_1 \\ \text{to get filter!} \end{array} \right.$