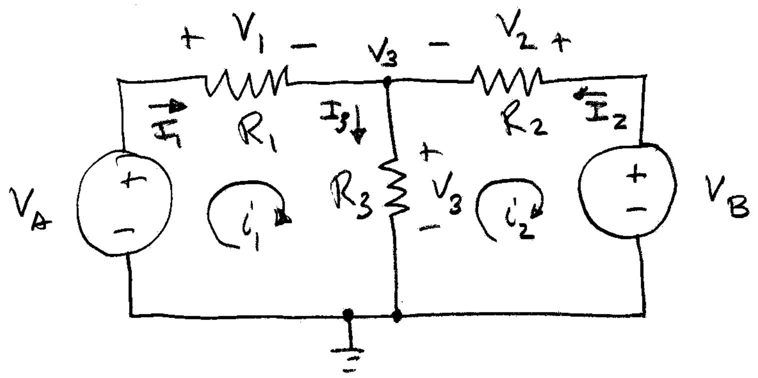
10/9/09 | Physics 116A | Mesh current example + Thévenin



Find I_3, V_3.

Planar circuit

use mesh current analysis.

Define mesh currents i_1, i_2 as shown.

Note $I_1 = i_1$, $I_2 = -i_2$, $I_3 = i_1 - i_2$.

Use KVL for the two loops:

$$V_A = V_1 + V_3 \qquad\qquad V_3 + V_2 = V_B$$

In terms of mesh currents,

$$V_1 = i_1 R_1, \quad V_3 = (i_1 - i_2) R_3, \quad V_2 = -i_2 R_2.$$

Two equations in two unknowns:

$$\left.\begin{aligned} i_1 R_1 + (i_1 - i_2) R_3 &= V_A \\ (i_1 - i_2) R_3 - i_2 R_2 &= V_B \end{aligned}\right\} \Rightarrow \begin{aligned} (R_1 + R_3)\, i_1 - R_3 i_2 &= V_A \\ R_3 i_1 - (R_2 + R_3)\, i_2 &= V_B \end{aligned}$$

Solve using Cramer's rule:

$$A = \begin{bmatrix} R_1 + R_3 & -R_3 \\ R_3 & -(R_2 + R_3) \end{bmatrix}$$

$$\begin{aligned} \Delta &\equiv \det A \\ &= -(R_1 + R_3)(R_2 + R_3) + R_3^2 \\ &= -(R_1 R_2 + R_2 R_3 + R_1 R_3) \end{aligned}$$

$$i_1 = \frac{\begin{vmatrix} V_A & -R_3 \\ V_B & -(R_2 + R_3) \end{vmatrix}}{\Delta} \qquad\qquad i_2 = \frac{\begin{vmatrix} R_1 + R_3 & V_A \\ R_3 & V_B \end{vmatrix}}{\Delta}$$

$$i_1 = \frac{-V_A (R_2 + R_3) + V_B R_3}{-(R_1 R_2 + R_2 R_3 + R_1 R_3)} \qquad\qquad i_2 = \frac{V_B (R_1 + R_3) - R_3 V_A}{-(R_1 R_2 + R_2 R_3 + R_1 R_3)}$$

$$i_1 = \frac{(R_2 + R_3) V_A - R_3 V_B}{R_1 R_2 + R_2 R_3 + R_1 R_3} \qquad\qquad i_2 = \frac{R_3 V_A - (R_1 + R_3) V_B}{R_1 R_2 + R_2 R_3 + R_1 R_3}.$$

Check result:

$$I_1 = i_1 \qquad I_2 = -i_2$$

Due to symmetry, the result for I_2 should equal the result for I_1 if we swap V_A and V_B and also swap R_1 and R_2.

$$I_1 = \frac{(R_2+R_3)V_A - R_3 V_B}{R_1R_2 + R_2R_3 + R_1R_3} \Rightarrow \begin{array}{l} V_A \Leftrightarrow V_B \\ R_1 \Leftrightarrow R_2 \\ I_1 \Leftrightarrow I_2 \end{array}$$

$$\Rightarrow \quad I_2 \overset{?}{=} \frac{(R_1+R_3)V_B - R_3 V_A}{R_2R_1 + R_1R_3 + R_2R_3}$$

agree so OK so far.

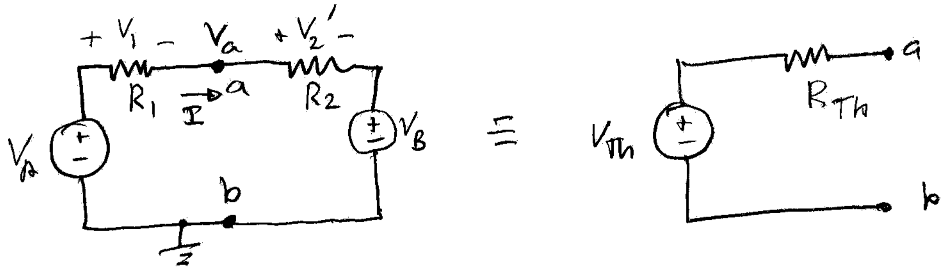
$$I_2 = -i_2 = \frac{(R_1+R_3)V_B - R_3 V_A}{R_1R_2 + R_2R_3 + R_1R_3}$$

Find desired results:

$$\begin{cases} I_3 = i_1 - i_2 = \dfrac{(R_2+R_3)V_A - R_3V_B}{R_1R_2+R_2R_3+R_1R_3} + \dfrac{(R_1+R_3)V_B - R_3V_A}{R_1R_2+R_2R_3+R_1R_3} \\ \quad = \dfrac{R_2V_A + R_1V_B}{R_1R_2+R_2R_3+R_1R_3} \\ V_3 = I_3R_3 = \dfrac{R_2R_3V_A + R_1R_3V_B}{R_1R_2+R_2R_3+R_1R_3} \end{cases}$$

Another approach:

Find Thévenin equivalent of circuit driving R_3 then use it to find I_3 and V_3.



Find open circuit voltage: $V_a \equiv V_{ab} = \text{open circuit voltage} = V_{Th}$

$$V_A = V_1 + V_2' + V_B = IR_1 + IR_2 + V_B$$

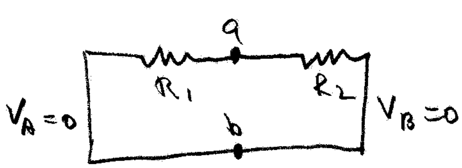
$$I = \frac{V_A - V_B}{R_1 + R_2}, \quad V_a \equiv V_{Th} = V_B + V_2' = V_B + IR_2$$

$$= V_B + \frac{V_A - V_B}{R_1 + R_2} R_2$$

$$= \frac{V_B R_1 + \cancel{V_B R_2} + V_A R_2 - \cancel{V_B R_2}}{R_1 + R_2}$$

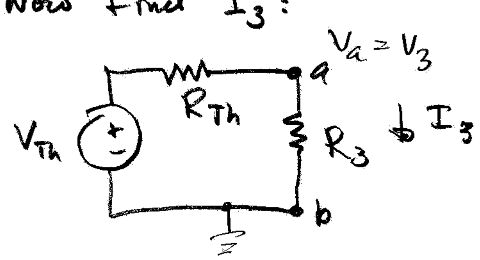
$$V_{Th} = \frac{R_2 V_A + R_1 V_B}{R_1 + R_2}$$

Find Thévenin equiv. resistance by setting independent sources to 0 and finding resistance of network between a and b:



$$R_{Th} = R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

Now find I_3:



$$I_3 = \frac{V_{Th}}{R_{Th} + R_3} = \frac{\dfrac{R_2 V_A + R_1 V_B}{R_1 + R_2}}{\dfrac{R_1 R_2}{R_1 + R_2} + R_3}$$

$$I_3 = \frac{R_2 V_A + R_1 V_B}{R_1 R_2 + R_1 R_3 + R_2 R_3} \quad \text{(as before)}$$

and $$V_3 = I_3 R_3 = \frac{R_2 R_3 V_A + R_1 R_3 V_B}{R_1 R_2 + R_2 R_3 + R_3 R_1}.$$