Poles and Zeros of $H(s)$, Analog Computers and Active Filters

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LRC Filter Poles and Zeros

- Pole structure same for all three functions (two poles)
- $H_R$ has two poles and zero at $s=0$: bandpass filter
- $H_L$ has two poles and double zero at $s=0$: high-pass filter
- $H_C$ has two poles and no zeros: low-pass filter

In terms of $\omega_r = \frac{1}{\sqrt{LC}}$
and $Q = \frac{\omega_r L}{R}$

\[ H_R(s) = \frac{V_R}{V_{in}} = \frac{i_R}{i_{in}} = \frac{R}{R + sL + \frac{1}{sC}} = \frac{RCs}{s^2LC + RCs + 1} = \frac{s/Q\omega_r}{(s/Q\omega_r)^2 + s/Q\omega_r + 1} \]
\[ H_L(s) = \frac{V_L}{V_{in}} = \frac{i_L}{i_{in}} = \frac{Ls}{R + sL + \frac{1}{sC}} = \frac{s^2LC}{s^2LC + RCs + 1} = \frac{(s/Q\omega_r)^2}{(s/Q\omega_r)^2 + s/Q\omega_r + 1} \]
\[ H_C(s) = \frac{V_C}{V_{in}} = \frac{1/sC}{2i_{in}} = \frac{1}{sC} \cdot \frac{1}{R + sL + \frac{1}{sC}} = \frac{1}{s^2LC + RCs + 1} = \frac{1}{(s/Q\omega_r)^2 + \frac{s}{Q\omega_r} + 1} \]
LRC Filter Pole Locations

Find pole locations: roots of denominator

\[ s^2LC + RC + 1 = 0 \]

\[ \omega_r = \frac{1}{RC}, \quad Q = \frac{\omega_r}{R} \]

\[ s = -\frac{R}{2L} \pm \frac{1}{2L} \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}} \]

\[ \omega = \frac{\omega_r}{2Q} \pm \sqrt{(\frac{\omega_r}{2Q})^2 - \omega_r^2} \]

3 cases:

1. \((\frac{\omega_r}{2Q})^2 < \omega_r^2\) complex conjugate roots in negative half-plane
2. \((\frac{\omega_r}{2Q})^2 = \omega_r^2\) double root on negative real axis \(s = -\frac{\omega_r}{2Q}\)
3. \((\frac{\omega_r}{2Q})^2 > \omega_r^2\) two roots on negative real axis

Poles at \(s_1\) and \(s_2\). If poles are distinct, the natural response is of form

\[ V_{\text{out}}(t) = A_1e^{s_1t} + A_2e^{s_2t} \]
The transfer function \( H(s) = \frac{V_{\text{out}}}{V_{\text{in}}} \) for the above LRC low-pass filter circuit is:

\[
H(s) = \frac{1}{1 + (s/\omega_0)^2 + s/(Q\omega_0)}
\]

where

\[
\omega_0 = \frac{1}{(LC)^{1/2}}, \quad Q = \frac{1}{R} \cdot \frac{L}{C}^{1/2}.
\]

* You can work with \( s \) directly as in Sec. 5.3 of Bobrow, or you can find \( H(j\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} \) using network analysis with complex impedance, then substitute \( s \) for \( j\omega \).
According to the given data:

\[ H(s) = \frac{1}{1 + (s/5)^2 + s/25} \]

\[ s_1 = -1/2 + j(3/2)(11)^{1/2}, \quad s_2 = -1/2 - j(3/2)(11)^{1/2} \]
From H(s) to V(t) via Laplace Transform

The linearity of the Laplace transform and its transformation of ordinary differential equations with constant coefficients to algebraic equations allows a direct connection between H(s) and f(t). Essentially, f(t), the inverse Laplace transform of H(s), represents the output voltage of the circuit for a unit impulse function (delta function) input. This also assumes the voltages and currents are zero at t=0 (zero initial conditions). The Laplace transform method is studied in 116B.

For this circuit, the inverse Laplace transform for H(s) is a damped sinusoid (see Table 5.1 in Bobrow or use Mathematica):

\[ V(t) = \frac{50}{3 \cdot 11^{1/2}} \exp(-t/2) \sin((3 \cdot 11^{1/2})t/2) u(t) \]

where \( u(t) \) is the unit step function.

Pole location determines angular frequency and rate of decay

Note that the expression is a linear combination of \( \exp(s_1 t) \) and \( \exp(s_2 t) \)
Critically Damped Case ($Q=0.5$)

\[ \omega_0 = 5 \quad H(s) = \frac{1}{1 + \left(\frac{s}{5}\right)^2} + \frac{s}{2.5} \]

The denominator has a double root at $s=-5$, leading to a double pole.

We see low pass filter behavior along the $j\omega$ axis.
Now, the inverse Laplace transform for $H(s)$ is:

$$V(t) = 25 \cdot t \cdot \exp(-5t) \cdot u(t)$$

as expected for the pole of order 2 (i.e., double root of denominator) on the negative real axis. The pole location determines the time constant of the exponential.

Again, this represents the output voltage of the circuit for a unit impulse function (delta function) input and zero initial conditions.

Pole location determines angular frequency ($\omega_0$) and rate of decay.
General Case of Poles and Zeros

(Following Bobrow, Sec. 5.4)

Circuit described by linear differential equation with constant, real coefficients

\[ a_n \frac{d^ny}{dt^n} + a_{n-1} \frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 \]

\[ = b_m \frac{d^mx}{dt^m} + b_{m-1} \frac{d^{m-1}x}{dt^{m-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 \]

with output (forced response) \( y(t) = Ye^{st} \) and input (forcing term) \( x(t) = Xe^{st} \).

Substituting the exponentials for \( x \) and \( y \) gives

\[ (a_ns^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0)Ye^{st} \]

\[ = (b_ms^m + b_{m-1}s^{m-1} + \ldots + b_1s + b_0)Xe^{st} \]

so the transfer function is a ratio of polynomials with real coefficients

\[ H(s) \equiv \frac{Y}{X} = \frac{b_ms^m + b_{m-1}s^{m-1} + \ldots + b_1s + b_0}{a_ns^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0}. \]
Pole and Zero Locations

It follows from the “fundamental theorem of algebra” that the polynomials can be factored into products of polynomials with real coefficients of order 1 or 2. Thus, we can write

\[ H(s) = \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} \]

where \( z_i \) and \( p_i \) are real or complex conjugate pairs.

- The \( z_i \) are the points where \( H(s) = 0 \) (zeros of the function)

- The \( p_i \) are the points where the denominator equals 0 (poles of \( H(s) \)). There can be repeated roots so the poles need not be simple poles, as we saw before.

- Natural response in the case of distinct poles is of form
  \[ y_{\text{nat}}(t) = A_1e^{p_1t} + A_2e^{p_2t} + \ldots + A_ne^{p_nt} \]
Synthesis of Network Using Op-Amps

- Can implement network with transfer function $H(s)$ using integrators, amplifiers and summing amplifiers

  - Example: $H(s) \equiv \frac{Y}{X} = (1 + \frac{s}{25} + \frac{s^2}{25})^{-1}$

  - Rewrite as $Y = (\frac{25}{s^2})X - (\frac{25}{s^2})Y - (\frac{1}{s})Y$

  - Recall for op-amp integrator, $H(s) = -\frac{1}{(RC)s}$

  - Use 4 integrators, two amplifiers to multiply by constants and a summing amplifier to implement function (perhaps not very efficient since we started with a second order differential equation...)

  - This is an analog computer, showing operational amplifiers in use for their original function: mathematical operations
Integrator and Network Block Diagram

Integrator:

\[ U_{in} \rightarrow \frac{C}{R} \rightarrow V_{out} \]

Virtual ground:

\[ V_{out} = -\frac{V_{in}}{RC} \]

\[ H(s) = \frac{V_{out}}{U_{in}} = -\frac{1}{RC} \]

Block diagram for network with:

\[ H(s) = \frac{Y}{X} = \frac{Y}{X} = \frac{1}{1 + \frac{s}{25} + \frac{s^2}{25}} \]

\[ Y = \frac{25}{8} X - \frac{25}{8} X - \frac{1}{2} Y \]

Integrator with:

\[ RC = \frac{1}{5} \text{ second} \]
Why Use Integrators?

• Could in principle use differentiators—

  • Switch R and C in op-amp circuit, get \( H(s) = -RC \, s \)

• \(|H(\omega)|\) increases with \(\omega\) so performance is sensitive to high frequency noise and amplifier instabilities (for reasons to be discussed later)

• Amplifier more likely to be overloaded by rapidly changing waveforms

• Initial conditions easier to set for integrator by switching a constant voltage across each integrating capacitor at \(t=0\)

• Thus, integrators are preferred
Analysis: start by defining the second derivative of $V$ at point $A$. Then just follow through the circuit tracing the paths and the rest follows.

Output at $D$ is proportional to $V$ and is a low pass filter. Output at $B$ is a band pass filter and at $A$ a high pass. Do you see why?

Can get poles in right half of $s$ plane by connecting $R'$ from $B$ to -Sum2 (NG)

Reference: Millman and Grabel, Microelectronics, Sec. 16-7
State Variable Filter Outputs

\[ H(s) = \frac{V_D}{V_{in}} = \frac{\omega_0^2 V}{V_{in}} = \frac{\omega_0^2}{s^2 + \omega_0^2} \]

\[ = \frac{1}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \]

Low pass (like \( H_L \))

\[ H(s) = \frac{V_Q}{V_{in}} = -\frac{\omega_0}{V_{in}} \frac{dV}{dt} = -\frac{\omega_0 \xi V}{V_{in}} \]

\[ = \frac{-\frac{\xi}{\omega_0}}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \]

Band-pass (like \(-QH_R\))

\[ H(s) = \frac{V_A}{V_{in}} = \frac{1}{V_{in}} \frac{d^2 V}{dt^2} = \frac{1}{V_{in}} \omega_0^2 V \]

\[ = \frac{\omega_0^2 \xi}{\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1} \]

High Pass (like \( H_A \))
Improved active filters: (Low pass example)

Start with RC low pass:

\[
\begin{align*}
\text{Transfer func} & \\
H(s) &= \frac{V_{out}}{V_{in}} \\
&= \frac{1}{sC} \\
&= \frac{\omega_c}{s + \omega_c} \\
\end{align*}
\]

Define \( A_v = |H(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}} \)

\( \omega_c \) is the half-power point (-3dB point)

and for \( \omega \to \omega_c \), \( A_v \sim \frac{1}{\omega} \) (falls at 20dB/decade)
Simple-Minded Approach: Buffered RC

We could get response with faster rolloff above \( \omega_c \) by cascading many such filters, buffered with amplifiers to prevent interactions with other stages due to load impedances:

\[
\omega_c = \frac{1}{RC}, \quad A_{u_0} = 1 + \frac{R_2}{R_1}
\]

(arbitrary)

\[
\frac{A_u}{A_{u_0}} = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}
\]

An RC Filter stage.

And this

\[
\text{3 pole RC Filter}
\]

would give a 60 dB/decade rolloff above \( \omega_c \)

(OK up to a few MHz with suitable op-amps)

But the performance near \( \omega_c \) can be improved by tailoring the locations of the poles.
Butterworth Filter Gives Sharper Cutoff

For example, the Butterworth filter has a sharper cutoff near \( \omega_c \) than a cascaded RC filter with the same number of poles while maintaining a smooth \( A_v(\omega) \) in the "passband." The denominator of \( H(s) \) consists of a Butterworth polynomial of degree \( n \) for an \( n \)-stage filter:

\[
\frac{H(s)}{A_v} = \frac{1}{B_n(s)}
\]

where

- \( n \)
- \( B_n(s) \) [Butterworth Polynomials]

\[
\begin{align*}
1 & \quad (s/\omega_c) + 1 \quad \text{(RC 1-pole filter)} \\
2 & \quad (s/\omega_c)^2 + 1 \quad \text{(2-pole filter)} \\
3 & \quad (s/\omega_c + 1)(s/\omega_c)^2 + \frac{1}{s/\omega_c + 1} \\
4 & \quad ((s/\omega_c)^2 + 0.765s/\omega_c + 1)((s/\omega_c)^2 + 1.848s/\omega_c + 1) \\
\text{etc.}
\end{align*}
\]
Implementation

\[ |B_n(j\omega)| = \sqrt{1 + (\frac{\omega}{\omega_n})^{2n}} \]

[Exercise: verify this for the case \( n=2 \)]

The 2nd order polynomial terms can be implemented as follows:

\[ \frac{H(s)}{A_{v_0}} = \frac{1}{(s/\omega_c)^2 + 2k(s/\omega_c) + 1} \]

where 2k is chosen to match the coefficient of \( s/\omega_c \) in the quadratic.

For op-amp, \( A_{v_0} = “\text{midband gain}” = 1 + \frac{R}{R’} \)

(“midband” means \( \omega \to 0 \) here.)
Analysis: Part I

\[ V_i = V_{out} \left( \frac{R_1}{(R_1 + R'_1)} \right) = \frac{V_{out}}{A_{V_0}} \]
where \( A_{V_0} = 1 + \frac{R'}{R_1} \)

Find \( H(s) = \frac{V_{out}}{V_m} \): (Express things in terms of \( V_{out} \) if possible)

\[ V_i = I \times \frac{1}{SC} \Rightarrow I = SC \frac{V_i}{V_{out}} \]

\[ V' = I \left( R + \frac{1}{SC} \right) = SC \frac{V_i}{V_{out}} \left( R + \frac{1}{SC} \right) = SC \left( R + \frac{1}{SC} \right) \frac{V_{out}}{V_{out}} \]

\[ = \left( 1 + \frac{RC}{SC} \right) \frac{V_{out}}{A_{V_0}} \]

\[ I_1 + I_2 = I \]

\[ \frac{V_m - V'}{R} + \frac{V_{out} - V'}{\frac{1}{SC}} = \frac{3 C V_{out}}{A_{V_0}} \]
Analysis: Part II

\[ \frac{V_{in}}{R} - \frac{(1 + RCS) V_{out}}{RA_{v0}} + SC V_{out} - \frac{SC(1 + RCS)}{A_{v0}} V_{out} \approx \frac{SC V_{out}}{A_{v0}} \]

\[ \frac{1}{R} - \frac{1 + RCS}{RA_{v0}} H(c) + \frac{SC H(c)}{A_{v0}} - \frac{SC(1 + RCS)}{A_{v0}} H(c) = \frac{SC}{A_{v0}} H(c) \]

\[ H(c) = \frac{SC}{A_{v0}} H(c) \]

\[ H(c) = \frac{SC}{A_{v0}} \left[ RCS + 1 + RCS + RCS + (RCS)^2 - RCS A_{v0} \right] \]

\[ H(c) = \frac{1}{A_{v0}} \left[ (RCS)^2 + (3 - A_{v0}) RCS + 1 \right] \]

Choose \( RC = \frac{1}{wc} \) and \( (3 - A_{v0}) = 2K \) to get desired filter stage.

i.e., \( 2K = 3 - A_{v0} = 2 - R_1/R_2 \Rightarrow R_1/R_2 = 2 - 2K \).

For \( n = 2 \), \( 2K = \sqrt{2} \), for example.
Other Filter Types and Comparisons

• The circuit can be modified to give high pass or bandpass filters (for example, the bandpass filter you made in lab)

• Can replicate any denominator function using proper choices of R, C and enough terms

• Another popular choice is Chebyshev filter - uses Chebyshev polynomials (specify corner freq., allowable passband variation)
  • Gives even sharper cutoff but has more ripple in the passband response

• Bessel filter is not as sharp as Butterworth near the cutoff frequency but has smoother phase response and less overshoot in time response with step function input

• Reference: Horowitz and Hill, The Art of Electronics, Ch. 5. See this book for comparisons and construction information.