**ENERGY** This is a concise and hopefully not too abstract summary of the ideas in Chapters 6 and 7.

**1. Work Causes Change in K** We accept that for a given object, $\Sigma \vec{F} = m\vec{a}$ . Now, whether or not it's obvious why, we can "dot" each side of this vector equation with the infinitesimal displacement $d\vec{r}$, then do a line integral along an arbitrary path from an initial location to a final.

$$\Sigma \int_i^f \vec{F} \cdot d\vec{r} = \int_i^f m\vec{a} \cdot d\vec{r} \qquad (1)$$

The right–hand side can be rewritten in the following way:

$$\int_i^f m\vec{a} \cdot d\vec{r} = m\int_i^f \frac{d\vec{v}}{dt} \cdot d\vec{r} = m\int_i^f d\vec{v} \cdot \frac{d\vec{r}}{dt} = m \int_i^f d\vec{v} \cdot \vec{v} = m \int_i^f (dv_x\, v_x + dv_y\, v_y + dv_z\, v_z)$$

$$= m\frac{v_x^2}{2}\Big|_i^f + m\frac{v_y^2}{2}\Big|_i^f + m\frac{v_z^2}{2}\Big|_i^f = \frac{1}{2}m\,(v_x^2 + v_y^2 + v_z^2)\Big|_i^f = \frac{1}{2}m\,v_f^2 - \frac{1}{2}m\,v_i^2 = \Delta\left(\frac{1}{2}mv^2\right)$$

Now, when a force $\vec{F}$ acts on an object, we *define* $\int_i^f \vec{F} \cdot d\vec{r}$ as the **work** done on the object by that force. Therefore, the *left*–hand side of equation (1), a sum, is the *net* work done by all forces. We also *define* $\frac{1}{2}mv^2$ as an object's **kinetic energy** K, the energy of motion, which, sensibly, is zero if v = 0. Therefore, the right–hand side of equation (1) is the change in kinetic energy.

$$W_{net} = \Delta(\tfrac{1}{2}\, m\, v^2) = \Delta K \qquad (2)$$

Note: Both sides of equation (1), being dot products, are scalars. Work and energy have no direction. This form of the "work–energy theorem" is nice, but it's not the one that we'll usually find most useful.

**2. Work by Nonconservative Forces Causes Change in Total Mechanical Energy** Forces can be divided into two classes: conservative, and nonconservative. It is a law of physics that energy is never *not* conserved, so the term "nonconservative" is misleading. A nonconservative force is one that doesn't conserve *a particular kind* of energy. But right now, what's more important is the definition of a *conservative* force. It is one for which $\int_i^f \vec{F} \cdot d\vec{r}$, i.e., the work done by that force, is the same for *any* path from i to f. Logically, the integral/work along any path "backward" to the starting point would be merely of the opposite sign, so the work for any *complete* path, ending where it started, is *zero* (which is an equivalent way of defining a conservative force).

Now think about this. If the integral is the same to get from i to f no matter how we get there, then we could define a value of some property related to this force at point i, and a value of that property at point f which differs from the value at point i by the integral. We couldn't do this if the integral weren't independent of the path. For instance, suppose we define the value of the property at point i to be 3.5. If we add to this the integral, we might get 3.5 + 1.2 if integrated along one path, but 3.5 + 6.8 if integrated along another; we couldn't define a unique value of the property at point f. But we can if the integral is independent of path. As it turns out, we find it most convenient to define a property whose difference from i to f is the *negative* of the integral, and we call this property **potential energy**, given the symbol U.

For a conservative force: $\quad \Delta U = U_f - U_i \equiv -\int_i^f \vec{F} \cdot d\vec{r} = -\text{Work} \qquad (3)$

Consider gravity. If a ball falls, gravity does positive work on it––$\vec{F}$ is in the direction of $d\vec{r}$, so the dot product is positive. But being lower, the ball has less ability to do work than before––it has less potential energy. Thus, when gravity does positive work, the potential energy *change* is negative. Conversely, if the ball rises, gravity does *negative* work and the potential energy change is *positive*.

Now, the actual potential energy function U depends on the nature of the force. In Physics 9A the two conservative forces we deal with most are gravity, where it's mgy, and a spring, where it's $\frac{1}{2}kx^2$. But in general we can say that $W = -\Delta U$ for each conservative force, so equation (2) becomes

$$W_{net} = W_{nonconservative} + \Sigma\, W_{conservative} = W_{nonconservative} + \Sigma\,(-\Delta U) = \Delta K \quad \text{or} \quad W_{nonconservative} = \Delta K + \Sigma\, \Delta U$$

Lumping all potential energies together as total potential energy U then gives us a famous form

$$W_{nonconservative} = \Delta K + \Delta U \qquad (4)$$

An important corollary to this is that if there *aren't* any significant nonconservative forces at play, then the **total mechanical energy**, K + U, is a constant. We'll use this a lot!

**3. Why Energy?** The difference between equations (2) and (4) is that in (2) we say all forces do work, and the result is $\Delta K$, which is true. But when we use equation (4), we *don't* explicitly concern ourselves with the work done by certain kinds of forces––conservative forces. Instead, by "moving them to the other side of the equation", the effect they have on the system is taken fully into account by referring to the potential energy associated with each such force. And for each of these, we never need concern ourselves with a tedious work integral, for it doesn't depend on the details of how the object gets from i to f. Besides being a profound concept, energy is a very powerful tool, for *we can often solve problems via energy a lot easier than via F = ma, because (1) energy is a scalar, so components, etc. don't come into play, and (2) we can often just look at i and f, and ignore the details of what happens in between.*

Now, if there *are* nonconservative forces at play, the work integral must be done, for it does depend on how the object gets from i to f. But the really easy cases are when there are no nonconservative forces, or when there are, but they do no work, perhaps being always perpendicular to $d\vec{r}$. Then, no matter how violent the accelerations, how much the forces may change in magnitude and direction, $K_f + U_f = K_i + U_i$.

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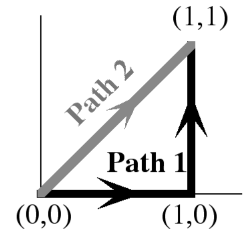
**4. The Mathematics of a Conservative Force** In Math 21 you'll go through the proof of the following:

$$\int_i^f \vec{F}\cdot d\vec{r} \text{ being independent of path} \Leftrightarrow \vec{F} = \vec{\nabla}A(x,y,z) \qquad (5)$$

The integral will be independent of path if and only if the force is the gradient of some scalar function $A(x,y,z)$ of position. A gradient is a kind of three–dimensional derivative, and it's natural to say: "Well, isn't this always the case? Couldn't any force $\vec{F}(x,y,z)$ be shown to be the derivative of some other function, just by integrating it?" This can fail in two ways: (1) Some forces *can't* be written as a function of position. Friction and air resistance aren't functions of *where* you are, but of which way you're going, and, in the case of air resistance, how fast. They depend not on position, but on velocity, either direction (friction) or magnitude and direction (air resistance). There is no function $A(x,y,z)$ that depends on *position* of which you could take a derivative and get something else, i.e., an F, that depends only on *velocity*! It's no coincidence that friction is nonconservative. If you slid a book along a table along the two paths shown, the work done by friction on Path 1 is small and negative, pushing always against the motion over a small distance, but along Path 2 it's huge and negative, pushing always against the motion over a huge distance. It's impossible to assign a potential energy difference if it depends on how you get from one place to another! (2) If the force is a function of position, in one dimension it might seem okay––you could just integrate $F(x)$ to find $A(x)$––but we live in a 3D world, and it's just not that easy in 3D. It may be a function of position, but still not the *gradient* of a function $A(x,y,z)$.


Consider the function $\vec{F} = (x+y)\,\hat{\imath} - (x+y)\,\hat{\jmath}$. Sure looks like you could integrate this. Well, you can, but the result depends on how you get from i to f, and, not coincidentally, there is no $A(x,y,z)$ of which it is the gradient. Let's not stumble around trying to prove that there's no such $A(x,y,z)$. But it's worthwhile to show that the integral/work is path–*dependent*. Consider the integral from (0,0) to (1,1) along Paths 1 and 2. Note that, since we don't move in the z–direction, $\int\vec{F}\cdot d\vec{r} = \int F_x\,dx + \int F_y\,dy$, and that along the first leg of Path 1, $y = 0$ and $dy = 0$, and along the second leg, $x = 1$ and $dx = 0$, and that all along Path 2, $x = y$.




Path 1: $\int F_x\,dx + \int F_y\,dy = \int_{x=0}^{x=1}(x+0)\,dx + \int_{y=0}^{y=1} -(1+y)\,dy = -1$

Path 2: $\int F_x\,dx + \int F_y\,dy = \int_{x=0}^{x=1}(x+x)\,dx + \int_{y=0}^{y=1} -(y+y)\,dy = 0$ Not the same!

Now let's make a seemingly small change in F, an itty–bitty sign. $\vec{F} = (x+y)\,\hat{\imath} + (x+y)\,\hat{\jmath}$. Consider it an exercise to show that this is *now* the gradient of a scalar function, $A(x,y,z) = \tfrac{1}{2}(x+y)^2$. Rather than bothering to show that the integral/work is now the same for *both* the above paths (though it might be good practice for you to show it), let's show it's the same for *all* paths! Using the definition of the gradient,

$$\int_i^f \vec{F}\cdot d\vec{r} = \int_i^f \vec{\nabla}A\cdot d\vec{r} = \int_i^f \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right) = \int_i^f dA = A(x_f,y_f,z_f) - A(x_i,y_i,z_i) = A_f - A_i = \Delta A$$

But wait; what does this have do with our present $\vec{F}$? Actually, it's general. It's "half" of the if–and–only–if in equation (5). *If indeed* $\vec{F} = \vec{\nabla}A(x,y,z)$, then the integral is independent of path. There's no need to grind through the details over and over again. The integral––the work––is always just $A_f - A_i$. The last thing to point out is that the function $A(x,y,z)$ of which $\vec{F}$ is the gradient is the negative of what we call the potential energy. Thus, work $= \int\vec{F}\cdot d\vec{r} = \Delta A = -\Delta U$, as in equation (3), and $\vec{F} = \vec{\nabla}A(x,y,z) = -\vec{\nabla}U(x,y,z)$.