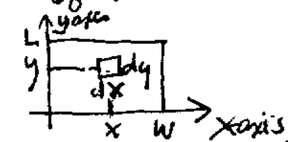
Integrating functions over areas and volumes in various coordinate systems

Rectangular coordinates (area of rectangle, width W, length L)

$$\int_0^W dx \int_0^L dy \, f(x,y)$$

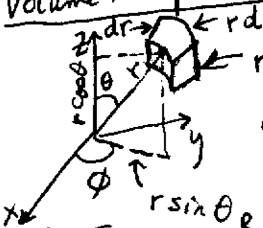


To integrate over rectangle, take function $f(x,y)$, multiply by area element $dx\,dy$ located at (x,y), and integrate x from 0 to W, integrate y from 0 to L.

Volume in Rectangular coordinates

$\int_0^W dx \int_0^L dy \int_0^H dz \, f(x,y,z)$ is an integral over a rectangular solid with width W, Length L, and height H.

Volume in Spherical coordinates (sphere radius R)



$$dV = dr(r\,d\theta)(r\sin\theta\,d\phi)$$

$$dV = r^2 dr \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^R r^2 dr \, f(r,\theta,\phi)$$

is the integral of f over the volume of the sphere.

If $f(r,\theta,\phi) = g(r)$, the integral becomes

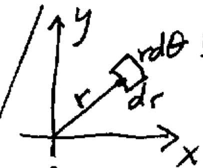
$$\underbrace{\int_0^{2\pi} d\phi}_{=2\pi} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{(2)} \int_0^R r^2 dr \, g(r) = \int_0^R r^2 dr \, g(r) = 4\pi \int_0^R r^2 dr \, g(r)$$

We could also get this volume element by taking the surface area of the sphere $4\pi r^2$, and multiplying by dr to get the volume of a spherical shell. Integrate from 0 to R to get integral over the volume of whole sphere.

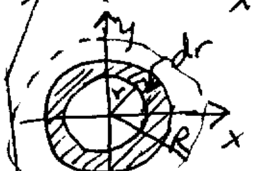
surface area of sphere $= 4\pi r^2$

$dV = 4\pi r^2 dr$

Polar coordinates (area of circle with radius R)



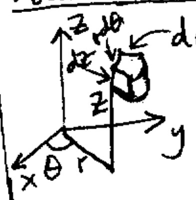
$$\int_0^{2\pi} d\theta \int_0^R r\,dr \, f(r,\theta)$$



If you are integrating a function $g(r)$ over a circle, $\int_0^{2\pi} d\theta \int_0^R r\,dr \, g(r) = 2\pi \int_0^R r\,dr\,g(r)$

i.e. Take circle of radius $2\pi r$, multiply by dr to get area of circular ring; Integrate $g(r)[2\pi r\,dr]$ between 0 and R to cover whole circle of radius R.

Volume in Cylindrical coordinates – Cylinder with radius R, height L



$dV = r\,d\theta\,dr\,dz$

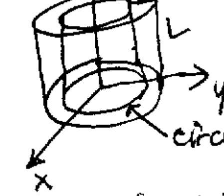
So to integrate $f(r,\theta,z)$ over a cylindrical volume in cylindrical coordinates, take

$$\int_0^{2\pi} d\theta \int_0^R r\,dr \int_0^L dz \, f(r,\theta,z)$$

If function $f(r,\theta,z)$ is separable, i.e., $f(r,\theta,z) = h(r)j(\theta)k(z)$, then the integral is $\int_0^{2\pi} d\theta \, j(\theta) \int_0^R r\,dr\,h(r) \int_0^L dz\,k(z)$.

If the function only depends on r, i.e., $j(\theta) = \text{const}$, $k(z) = \text{Const}$ then integral becomes $\int_0^{2\pi} d\theta \int_0^L dz \int_0^R r\,dr\,h(r) = 2\pi L \int_0^R r\,dr\,h(r)$

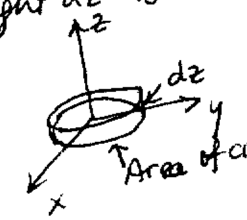
We could also get this by thinking about integrating over rings of area $2\pi r\,dr$ with height L, and then get the whole volume by integrating over dr from 0 to R; we are integrating over cylindrical shells, as shown.



circumference of circle is $2\pi r$.

If the function depends only on z, i.e. $j(\theta) = \text{const}$, $h(r) = \text{Const}$, then the integral is $\int_0^{2\pi} d\theta \int_0^R r\,dr \int_0^L dz\,k(z) = \pi R^2 \int_0^L dz\,k(z)$

Now we are integrating over circles with area πR^2 and height dz to make small cylindrical volumes (disks)



Area of circle is πR^2

Charge

$Q = \int \rho \, dV$ (ρ: volume charge density) Volume integral

$= \int \sigma \, da$ (σ: surface charge density) Area integral

$= \int \lambda \, dl$ (λ: linear charge density) line integral

Current

$I = \int \vec{J} \cdot d\vec{a}$ Surface integral