Maxwell's Equations

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{B} = \frac{4\pi\vec{J}}{c} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

Without sources ($\rho=0, \vec{J}=0$).

① $\vec{\nabla}\cdot\vec{E} = 0$

② $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$

③ $\vec{\nabla}\cdot\vec{B} = 0$

④ $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

Vector identities: $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$, where $\vec{A}$ = any vector

Definitions in Cartesian coordinates: $\vec{\nabla}\psi = \hat{x}\frac{\partial}{\partial x}\psi + \hat{y}\frac{\partial}{\partial y}\psi + \hat{z}\frac{\partial}{\partial z}\psi$

$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{y}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \hat{z}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)$$

Derivation of wave equation for $\vec{B}$: Take curl of eq ④: $\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{1}{c}\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}$

$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{0 \text{ from ③}} - \nabla^2\vec{B} = \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$ Substitute ② for $\vec{\nabla}\times\vec{E}$

$= \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$

$\nabla^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}$ or $\boxed{\nabla^2 B - \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} = 0}$ wave eq for $\vec{B}$, velocity c

12/4/06 Derivation of wave equation for $\vec{E}$: Take curl of eq ②. $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{1}{c}\vec{\nabla}\times\left(\frac{\partial\vec{B}}{\partial t}\right)$

$\vec{\nabla}\underbrace{(\vec{\nabla}\cdot\vec{E})}_{0 \text{ from ②}} - \nabla^2\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})$ Substitute ④ for $\vec{\nabla}\times\vec{B}$

$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$

$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$ or $\boxed{\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0}$ Wave equation for $\vec{E}$, wave has velocity c

For all waves, $\lambda f = c$, $\omega = 2\pi f$, $k = \frac{2\pi}{\lambda}$, $\therefore \omega = ck$

Plane wave solutions of wave equation: $\vec{E} = \vec{E}_{max}\sin(\omega t - kx)$

wave propagating along $\hat{x}$. $\vec{B} = \vec{B}_{max}\sin(\omega t - kx)$

The wave is transverse. Proof: $\vec{\nabla}\cdot\vec{E} = E_{max,x}\frac{\partial}{\partial x}[\sin(\omega t - kx)] = 0 \Rightarrow E_{max,x} = 0$

$\vec{E}\perp\vec{B}$. Proof: Take $\vec{E}_{max}$ to lie along $\hat{y}$, i.e. $\vec{E} = E_{max}\,\hat{y}\sin(\omega t - kx)$

Then $\vec{\nabla}\times\vec{E} = \hat{z}\frac{\partial}{\partial x}E_y = \hat{z}E_{max}(-k)\cos(\omega t - kx) = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{B}$ along $\hat{z}$

Let $\vec{B} = B_{max}\,\hat{z}\sin(\omega t - kx)$, $-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = -\frac{1}{c}B_{max}\,\hat{z}\,\omega\cos(\omega t - kx)$

$\Rightarrow E_{max}k = \frac{\omega}{c}B_{max}$; Since $k = \frac{\omega}{c}$, $\boxed{E_{max} = B_{max}}$

$\therefore \vec{E}_{max}\perp\vec{B}_{max}$, and $\hat{x}, \vec{E}, \vec{B}$ make up directions for right handed coordinate system. In fact $\underbrace{\vec{S}}_{\text{Poynting vector}} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{\text{energy}}{\text{area}\cdot\text{time}}$, energy flow of wave through area