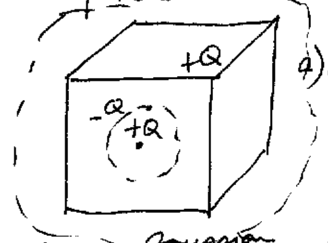
Physics 9HD - Midterm 1, 2007, Solutions

1\.



a) Induced charge on inside of spherical cavity = $-Q$

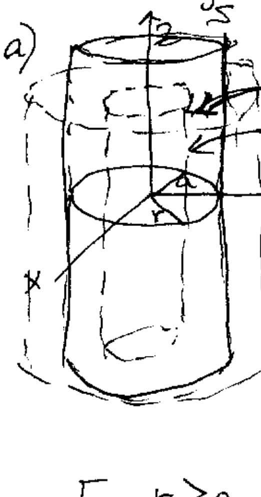
Induced charge on outside of cube = $+Q$ since cube is neutral.

$\vec{E}=0$ inside body of any perfect conductor, including this one.

b) Place a Gaussian surface of any shape outside the cube. Then Gauss's law says $\int_S \vec{E}\cdot d\vec{a} = 4\pi Q_{encl}$

$\therefore \quad \int_S \vec{E}\cdot d\vec{a} = \boxed{\text{total electric flux through surface} = 4\pi Q}$

2 a)



small Gaussian cyl. radius $r\le a$, length L

larger Gaussian cylinder, radius $r>a$, length L

$\rho = Cr$ for $r\le a$. For $r\le a$, Consider Gaussian cylinder of radius r, and length L. By symmetry, E is const on cyl. surf.

Gauss's law: $\int_S \vec{E}\cdot d\vec{a} = 4\pi Q_{encl}$

$$E(2\pi r L) = 4\pi\int_0^r \rho(r')(2\pi r' dr')L = 8\pi^2 L\int_0^r C r'^2 dr'$$

$$E(2\pi r L) = 8\pi^2 L C\frac{r^3}{3}$$

$$\boxed{\vec{E} = 4\pi C\frac{r^2}{3}\hat{r}}, \quad r\le a$$

For $r>a$, Consider larger Gaussian cylinder of radius $r>a$, length L

Gauss's law: $\int \vec{E}\cdot d\vec{a} = 4\pi Q_{encl}$.

$$E(2\pi r L) = 4\pi\int_0^a \rho(r')\, 2\pi r' dr' L = 8\pi^2 L\int_0^a C r'^2 dr'$$

$$E(2\pi r L) = 8\pi^2 L C\frac{a^3}{3}$$

$$r\ge a, \quad \boxed{\vec{E} = 4\pi C\frac{a^3}{3r}\hat{r}}$$

Potential for $r\le a$, $\varphi(r) - \underset{0}{\varphi(0)} = -\int_0^r \vec{E}\cdot d\vec{s} = -\frac{4\pi C}{3}\int_0^r r'^2 dr' = -4\pi C\frac{r^3}{9}$

$$\boxed{r\le a, \ \varphi(r) = -\frac{4\pi C r^3}{9}}, \quad \varphi(a) = -\frac{4\pi C a^3}{9}$$

$r>a$, Potential $\varphi(r) - \varphi(a) = -\int_a^r \vec{E}\cdot d\vec{s} = -4\pi C\frac{a^3}{3}\int_a^r\frac{dr}{r}$

$$\varphi(r) + \frac{4\pi C a^3}{9} = -\frac{4\pi C a^3}{3}\ln\frac{r}{a}$$

$$\boxed{r\ge a, \ \varphi(r)} = -\frac{4\pi C a^3}{9} - \frac{4\pi C a^3}{3}\ln\frac{r}{a} = \boxed{\frac{-4\pi C a^3}{3}\left(\frac{1}{3} - \ln\frac{r}{a}\right)}$$

b) $\nabla^2\varphi(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)$, since φ does not depend on angle ϕ or coordinate z.

$$r\le a, \quad \nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(-\frac{4\pi C r^3}{9}\right)\right) = -\frac{4\pi C}{\cancel{9}_3}\frac{1}{r}\frac{\partial}{\partial r}\left(r(\cancel{3}r^2)\right) = \frac{-4\pi C}{\cancel{3}r}(\cancel{3}r^2)$$

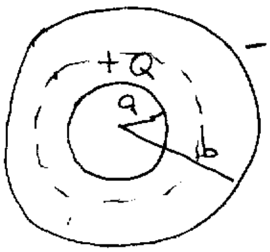
$$\boxed{r\le a \quad \nabla^2\varphi = -4\pi C r = -4\pi\rho(r)} \quad \text{Poisson's equation}$$

2b) $r \geq a$, $\nabla^2 \varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\left[-\frac{4\pi C a^3}{3}\left(\frac{1}{3} - \ln\frac{r}{a}\right)\right]\right)$

$= \frac{1}{r}\frac{\partial}{\partial r}\left(\cancel{r}\left(+\frac{4\pi C a^3}{3\cancel{r}}\right)\right) = 0$

$\boxed{r \geq a,\ \nabla^2\varphi = 0}$ Laplace's equation

3. a)



Place charge of $+Q$ on inner conductor, $-Q$ on outer conductor

For Gaussian sphere, $a \leq r \leq b$, $\int_S \vec{E}\cdot d\vec{a} = 4\pi Q$

$E(4\pi r^2) = 4\pi Q$ since E is constant over Gaussian sphere by symmetry

$a \leq r \leq b$, $\vec{E} = \frac{Q}{r^2}\hat{r}$ in spherical coordinates

Potential $\varphi(a) - \varphi(b) = -\int_b^a \vec{E}\cdot d\vec{s} = -\int_b^a \frac{Q}{r^2}dr = \left[\frac{Q}{r}\right]_b^a = Q\left(\frac{1}{a} - \frac{1}{b}\right)$

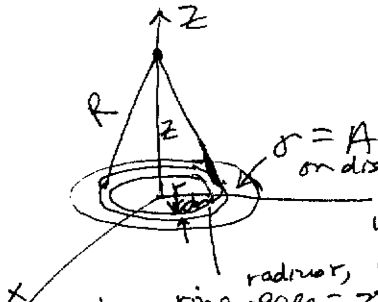
$C = \frac{Q}{V} = \frac{Q}{Q\left(\frac{1}{a}-\frac{1}{b}\right)} = \boxed{\frac{ab}{b-a}}$

b) $U = \int_V \frac{E^2}{8\pi}dV = \frac{1}{8\pi}\int_a^b \left(\frac{Q}{r^2}\right)^2 (4\pi r^2 dr) = \frac{Q^2}{2}\int_a^b \frac{dr}{r^2} = \frac{Q^2}{2}\left[\frac{-1}{r}\right]_a^b = \frac{Q^2}{2}\left(\frac{-1}{b} + \frac{1}{a}\right)$

$= \boxed{\frac{Q^2}{2}\left(\frac{1}{a} - \frac{1}{b}\right)}$

$\frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{ab}{b-a}\right)(Q^2)\left(\frac{1}{a} - \frac{1}{b}\right)^2 = \boxed{\frac{Q^2}{2}\left(\frac{1}{a} - \frac{1}{b}\right) = U}$

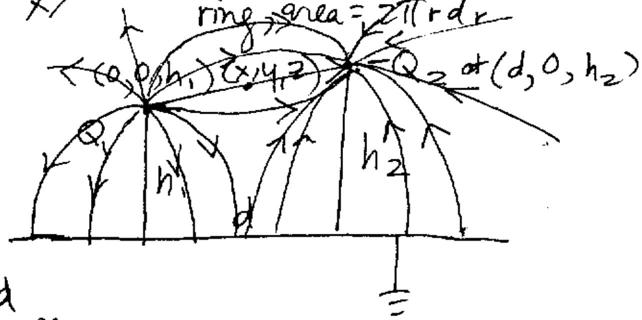
4.



$\boxed{\varphi(z)} = \int \frac{\sigma\, da}{R} = \boxed{\int_0^a \frac{Ar\,(2\pi r\,dr)}{\sqrt{r^2 + z^2}}} = 2\pi A \int_0^a \frac{r^2 dr}{\sqrt{r^2+z^2}}$

$\boxed{\vec{E}} = -\vec{\nabla}\varphi = -\hat{z}\frac{\partial}{\partial z}\varphi(z) = \boxed{-\hat{z}\left(2\pi z A\right)\int_0^a \frac{r^2\,dr}{(r^2+z^2)^{3/2}}}$

5.



$$\boxed{\begin{aligned}\varphi(x,y,z) = {} & \frac{Q_1}{\sqrt{x^2+y^2+(z-h_1)^2}} - \frac{Q_2}{\sqrt{(x-d)^2+y^2+(z-h_2)^2}} \\ & - \frac{Q_1}{\sqrt{x^2+y^2+(z+h_1)^2}} + \frac{Q_2}{\sqrt{(x-d)^2+y^2+(z+h_2)^2}}\end{aligned}}$$

$\boxed{\vec{E} = -\vec{\nabla}\varphi = -\hat{x}\frac{\partial\varphi}{\partial x} - \hat{y}\frac{\partial\varphi}{\partial y} - \hat{z}\frac{\partial\varphi}{\partial z}}$

Add 2 image charges below plane

$-Q_1$ at $(0,0,-h_1)$

$+Q_2$ at $(d,0,-h_2)$