

**Complex Numbers:** A complex number has two parts: a real number, and another real number multiplied by  $i$ , where  $i$  is defined by  $i^2 = -1$ . For example, in  $X = 3 + i 4$  the ‘real part’ is 3 and the ‘imaginary part’ (which is a real number) is 4. Complex numbers include real numbers as a subset, *i.e.* when the imaginary part is zero, and may be thought of as a compact way of conveying twice the information of a real number. However, they also obey special rules of arithmetic, based on keeping  $i$  (to the first power) separate, while  $i^2$  becomes  $-1$ :

$$\begin{aligned} \text{Given:} \quad U &= u_1 + i u_2 \quad \text{and} \quad V = v_1 + i v_2 \\ U + V &= (u_1 + i u_2) + (v_1 + i v_2) = (u_1 + v_1) + i (u_2 + v_2) \\ U V &= (u_1 + i u_2) (v_1 + i v_2) = u_1 v_1 + i u_1 v_2 + i u_2 v_1 + i^2 u_2 v_2 = (u_1 v_1 - u_2 v_2) + i (u_1 v_2 + u_2 v_1) \end{aligned}$$

The two parts of a complex number may each be functions, *e.g.*  $U(x) = u_1(x) + i u_2(x)$ .

A function whose *argument* is complex may often be (and usually is) broken into its real and imaginary parts. A common function is the exponential. Consider first an exponential whose argument has no *real* part:  $e^{ix}$ , where  $x$  is real. With the power series,  $e^b = \sum_0^\infty \frac{b^n}{n!}$ , we may write:

$$e^{ix} = \sum_0^\infty \frac{(ix)^n}{n!} = 1 + i x + i^2 \frac{1}{2!} x^2 + i^3 \frac{1}{3!} x^3 + i^4 \frac{1}{4!} x^4 + i^5 \frac{1}{5!} x^5 + i^6 \frac{1}{6!} x^6 + i^7 \frac{1}{7!} x^7 + i^8 \frac{1}{8!} x^8 + \dots$$

Using  $i^3 = i^2 i = -i$ ,  $i^4 = i^3 i = 1$ ,  $i^5 = i^4 i = i$ , *etc.*, this naturally breaks into two series.

$$e^{ix} = \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \frac{1}{8!} x^8 + \dots \right) + i \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \right)$$

The sums in parentheses are the power series for cosine and sine, and the result is an identity used very often in physics, known as the *Euler formula*:

$$e^{+ix} = \cos x + i \sin x$$

If the argument also has a real part then:  $e^X = e^{(x_1 + ix_2)} = e^{x_1} e^{ix_2} = (e^{x_1} \cos x_2) + i (e^{x_1} \sin x_2)$ .

Useful formulas following directly from the Euler formula are:

$$\cos x = \frac{e^{+ix} + e^{-ix}}{2} \quad \sin x = \frac{e^{+ix} - e^{-ix}}{2i}$$

In the same way as a two-dimensional vector (by which it can be represented and which shares the same addition rule), a complex number has a real positive magnitude defined as the square-root of the sum of the squares of its two parts.

$$|U| = \sqrt{u_1^2 + u_2^2}$$

Given a complex number or function  $X$ , its *complex conjugate*, denoted  $X^*$ , is obtained by replacing  $i$  by  $-i$  wherever it occurs. Examples:

$$\begin{aligned} U = u_1 + i u_2 &\Rightarrow U^* = u_1 - i u_2 & f(x) = \cos x - i \sin x &\Rightarrow f^*(x) = \cos x + i \sin x \\ f(x) = e^{-x^2} e^{+ix} &\Rightarrow f^*(x) = e^{-x^2} e^{-ix} & f(x) = \tanh^{-1}(3 - i 4) &\Rightarrow f^*(x) = \tanh^{-1}(3 + i 4) \end{aligned}$$

The product of a quantity and its complex conjugate gives the square of the quantity’s magnitude.

$$U^* U = (u_1 - i u_2)(u_1 + i u_2) = u_1^2 + u_2^2 = |U|^2$$

An important special function is the *complex exponential*  $e^{ikx} = \cos(kx) + i \sin(kx)$ . It is oscillatory because both its real and imaginary parts are oscillatory, of angular frequency  $k$ . (With  $x$  measured in meters,  $k$  would have to be in rad/m. An equivalent function is  $e^{i\omega t}$ , where  $\omega$  would be measured in rad/s.) Moreover, its magnitude is unity:

$$f(x) = e^{ikx} \Rightarrow f^*(x) f(x) = e^{-ikx} e^{+ikx} = e^{-ikx + ikx} = e^0 = 1$$

$$\text{or } f^*(x) f(x) = (\cos(kx) - i \sin(kx)) (\cos(kx) + i \sin(kx)) = \cos^2(kx) + \sin^2(kx) = 1$$

Thus, it rotates in the real-imaginary plane at frequency  $k$  and maintains unit length.

