Solution for Physics 108 Final (2015)

1 − (1) From $S_i + S_o = 60$ cm and

$$M = -\frac{S_i'}{S_o} \text{ (thin lens) } = -2$$

We have $S_i = 2S_o$. Thus

$$S_o = 20 \text{ cm}; \quad S_i = 40 \text{ cm}.$$

From $\frac{1}{S_o} + \frac{1}{S_i'} = \frac{1}{f}$ (on thin lens)

We have

$$f = \frac{S_o S_i'}{S_o + S_i} = \frac{40}{3} \text{ cm}$$

1 − (2) If using two identical lenses that are placed one right after another with f', then

$$\frac{1}{f'} + \frac{1}{f'} = \frac{1}{f} = \frac{2}{f'}$$

Thus,

$$f' = 2f = \frac{80}{3} \text{ cm}$$

\#

2-(1) Let the minimum resolved angle be $\delta\alpha_0$ in air. By Snell's law, it is related to the minimum resolved angle in a watery eye by

$$\delta\alpha_0 = n_w \delta\alpha_{eye} = n_w \cdot \frac{\lambda_0}{n_w d} = \frac{\lambda_0}{d}$$

Thus the minimum separation on an airplane at $L = 1000$ m away is

$$\delta l = L \cdot \delta\alpha_0 = \frac{\lambda_0 L}{d} = 0.1\,\text{m} = 10\,\text{cm}.$$

2-(2) If one uses a binocular with $m_{angle} = 50$, then the minim resolvable angle

$$\delta\alpha_0' = \frac{1}{50}\left(n_w \delta\alpha_{eye}\right) = \left(\frac{1}{50}\right)\frac{\lambda_0}{d}$$

Thus

$$\delta l' = \frac{1}{50}\,\delta l = 2\,\text{mm}$$

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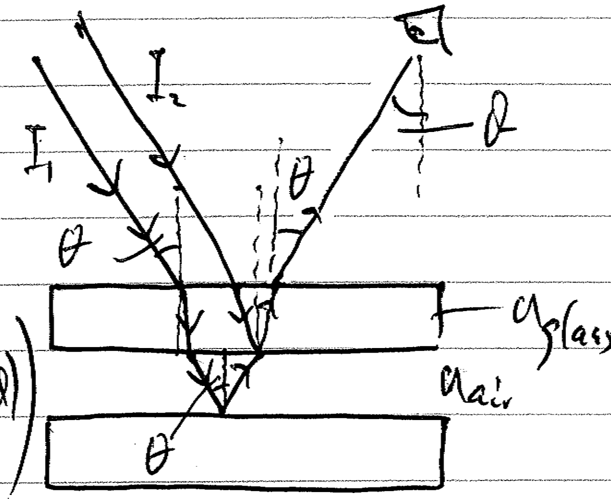
3-(1) As a function of viewing angle θ from the normal incidence,



$$I(\theta) = 2I_1\left(1-\cos\left(\frac{4\pi}{\lambda_0} d\cos\theta\right)\right)$$

Since the center is at $\theta_0 = 0$ and is completely dark,

$$I(0) = 2I_1\left(1-\cos\left(\frac{4\pi d}{\lambda_0}\right)\right) = 0$$

Thus

$$\frac{4\pi d}{\lambda_0} = 2\pi \cdot m$$

$$I(\theta) = 2I_1\left(1-\cos\left(2\pi \cdot m \cdot \cos\theta\right)\right).$$

3-(2) If at $\theta = 5^\circ = 0.0875$ radian

$$I(\theta) = 0 = 2I_1\left(1-\cos\left(2\pi \cdot m \cdot \cos\theta\right)\right)$$

then $m \cdot \cos\theta = m - 1$

$$m = \frac{1}{1-\cos\theta} = 263 \Rightarrow d = \frac{\lambda_0}{2} m = 79\ \mu m$$

4-(1) Since $n(\text{red}) < n(\text{yellow}) < n(\text{green}) < n(\text{blue}) < n(\text{violet})$, then from

$$\theta_{ic} = \sin^{-1}\left(\frac{1}{n}\right)$$

Violet is totally reflected first, and red is totally reflected last.

4-(2) Since ω_0 for water is larger than ω_0 for glass when ω increases from red to violet, $n(\omega) = \sqrt{\epsilon(\omega)}$ changes ~~less for water less for water than~~ less in water than in glass:

$$\left|\frac{d\epsilon(\omega)}{d\omega}\right| = \frac{2\omega\omega_p^2}{(\omega_0^2-\omega^2)^2} \propto \frac{1}{(\omega_0^2-\omega^2)^2}$$

Thus the angler spread of the colors in water is smaller than the spread in glass. Mathematically, from

$$\sin\theta_1 = n(\omega)\sin\theta_2(\omega)$$

For a fixed θ_1,

$$\delta(\sin\theta_1) = 0 = \delta n(\omega)\cdot\sin\theta_2 + n(\omega)\cos\theta_2\cdot\delta\theta_2$$

$$\therefore \left|\frac{d\theta_2}{d\omega}\right| = \left|\frac{d\theta_2}{dn}\cdot\frac{dn}{d\omega}\right|$$

$$= \underbrace{\frac{1}{n(\omega)}\tan\theta_2(\omega)}_{\sim \text{constant}} \cdot \left|\frac{dn}{d\omega}\right|$$

$$\propto \left|\frac{dn}{d\omega}\right| \quad \#$$

$$\because \left|\frac{dn}{d\omega}\right|_{\text{water}} < \left|\frac{dn}{d\omega}\right|_{\text{glass}}$$

$$\therefore \left|\frac{d\theta_2}{d\omega}\right|_{\text{water}} < \left|\frac{d\theta_2}{d\omega}\right|_{\text{glass}} \quad \#$$

5-(i) Let $n_1 < n_2$ to be specific, and $\theta_1 < \theta_{1B}$.

Since

$$R_s = |r_s|^2 = \left(\frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \right)^2 \quad (1)$$

$$R_p = |r_p|^2 = \left(\frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1} \right)^2 \quad (2)$$

Proving $R_s \geq R_p$ is equivalent to proving

$$(n_1 \cos\theta_2 + n_2 \cos\theta_1)^2 (n_1 \cos\theta_1 - n_2 \cos\theta_2)^2 \quad (3)$$

is larger than

$$(n_1 \cos\theta_1 + n_2 \cos\theta_2)^2 (n_1 \cos\theta_2 - n_2 \cos\theta_1)^2 \quad (4)$$

With $\theta_1 < \theta_{1B}$, we have both

$$n_2 \cos\theta_2 > n_1 \cos\theta_1 \quad (5)$$

$$n_2 \cos\theta_1 > n_1 \cos\theta_2 \quad (6)$$

Then we only need to prove

$$(n_1 \cos\theta_2 + n_2 \cos\theta_1)(n_2 \cos\theta_2 - n_1 \cos\theta_1) \cdots (7)$$

$$\geq (n_1 \cos\theta_1 + n_2 \cos\theta_2)(n_2 \cos\theta_1 - n_1 \cos\theta_2) \cdots (8)$$

Now

$$⑦ = u_1 u_2 (\cos^2\theta_2 - \cos^2\theta_1) + \cos\theta_1 \cos\theta_2 (u_2^2 - u_1^2)$$

$$⑧ = u_1 u_2 (\cos^2\theta_1 - \cos^2\theta_2) + \cos\theta_1 \cos\theta_2 (u_2^2 - u_1^2)$$

Since $\theta_1 > \theta_2$ $(u_1 < u_2)$, $\cos^2\theta_2 > \cos^2\theta_1$

$$u_1 u_2 (\cos^2\theta_2 - \cos^2\theta_1) > 0$$

$$u_1 u_2 (\cos^2\theta_1 - \cos^2\theta_2) < 0$$

$$\therefore ⑦ > ⑧ \Rightarrow R_s \geq R_p$$

5-(2) For $\theta_1 \geq \theta_{1B}$, proving $R_s \geq R_p$ is equivalent to proving

$$(u_1 \cos\theta_2 + u_2 \cos\theta_1)(u_2 \cos\theta_2 - u_1 \cos\theta_1) \cdots ⑦'$$

$$\geq (u_1 \cos\theta_1 + u_2 \cos\theta_2)(u_1 \cos\theta_2 - u_2 \cos\theta_1) \cdots ⑧'$$

$$⑦' = \cos\theta_1 \cos\theta_2 (u_2^2 - u_1^2) + u_1 u_2 (\cos^2\theta_2 - \cos^2\theta_1)$$

$$⑧' = \cos\theta_1 \cdot \cos\theta_2 (u_1^2 - u_2^2) + u_1 u_2 (\cos^2\theta_2 - \cos^2\theta_1)$$

Now since $(7)' > 0$, $(8)' > 0$, but

$$\cos\theta_1 \cos\theta_2 (a_2^2 - a_1^2) > 0$$

$$\cos\theta_1 \cos\theta_2 (a_1^2 - a_2^2) < 0,$$

we have

$$(7)' > (8)'$$

thus $R_s > R_p$ for $\theta_1 > \theta_{1B}$ #

6-(1) For $n_1 > n_2$,

$$\tan\theta_{1B} = \frac{n_2}{n_1} = \frac{\sin\theta_{1B}}{\cos\theta_{1B}}$$

$$\sin\theta_{1C} = \frac{n_2}{n_1}$$

$$\therefore \sin\theta_{1B} = \cos\theta_{1B} \cdot \left(\frac{n_2}{n_1}\right)$$

$$= \cos\theta_{1B} \cdot \sin\theta_{1C}$$

$$< \sin\theta_{1C}$$

since $\cos\theta_{1B} < 1$ ※

6-(2) When $\vec{E} \parallel OA$, we call it e-ray ($\vec{E}_e$), the critical angle for e-ray is

$$\theta_{1C}^{(e)} = \sin^{-1}\left(\frac{1}{n_e}\right) = 42.3^\circ$$

When $\vec{E} \perp OA$, we call it o-ray ($\vec{E}_o$), the critical angle for o-ray is

$$\theta_{1C}^{(o)} = \sin^{-1}\left(\frac{1}{n_o}\right) = 37.1^\circ.$$

7-(a) $$\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = 2 \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} = 2 \begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix}$$

Linearly polarized at an angle 60° from the x-axis

7-(b) $$\begin{pmatrix} -i \\ -1 \end{pmatrix} = \frac{\sqrt{2}}{i} \begin{pmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{pmatrix} = \left(-i\sqrt{2}\right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Right-circularly polarized.

7-(c) $$\begin{pmatrix} 1+i \\ -1+i \end{pmatrix} = \sqrt{2} \begin{pmatrix} e^{i\pi/4} \\ e^{i3\pi/4} \end{pmatrix} = \sqrt{2}\, e^{i\pi/4} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \left(2e^{i\pi/4}\right) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Left-circularly polarized.

7-(d) $$\begin{pmatrix} i \\ 1+i \end{pmatrix} = \begin{pmatrix} e^{i\pi/2} \\ \sqrt{2}\, e^{i\pi/4} \end{pmatrix} = \sqrt{3}\, e^{i\pi/2} \begin{pmatrix} 1/\sqrt{3} \\ \sqrt{2/3} \cdot e^{-i\pi/4} \end{pmatrix}$$

Elliptically polarized, ccw, *

8-(1) Simply a linear polarizer with TA ∥ y

$$M_{linear}(TA \parallel \hat{y}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$M_{linear}(\theta = 90^\circ) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}$$

Then with $\tilde{E}_{inc} = \begin{pmatrix} \alpha \\ \beta e^{i\Delta\phi_0} \end{pmatrix}$,

$$\tilde{E}_{out} = M_{linear}(TA \parallel \hat{y})\, \tilde{E}_{inc}$$

$$= \beta e^{i\Delta\phi_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The total energy is reduced from

$(\alpha^2 + \beta^2) = 1$

to $\beta^2 \leq \alpha^2 + \beta^2 = 1$ #

8-(2) Starting with a linear polarizer with TA at 45° from x-axis,

$$M_{Linear}(\theta=45^\circ) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

then the output from the linear polarizer

$$\tilde{E}^{(1)}_{out} = M_{Lin}(\theta=45^\circ)\,\tilde{E}_{in}$$

$$= \frac{1}{\sqrt{2}}\left(\alpha+\beta e^{i\Delta\phi_0}\right)\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Now for a left-circular polarized light, we need to add $+\pi/2$ phase to the y-component. We use a $\lambda/4$-plate so oriented that

$$M_{\lambda/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$\tilde{E}^{(2)}_{out} = M_{\lambda/4}\,\tilde{E}^{(1)}_{out} = \frac{1}{\sqrt{2}}\left(\alpha+\beta e^{i\Delta\phi_0}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\left|\frac{1}{\sqrt{2}}\left(\alpha+\beta e^{i\Delta\phi_0}\right)\right|^2 \leq 1$$

*

If you start from the answer of Part (1)

$$\tilde{E}_{out}^{(1)} = \beta e^{i\Delta\phi_0} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If we have a $\lambda/4$ plate set at $\theta = 45^\circ$, then

$$M_{\lambda/4}(\theta = +45^\circ) = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

then

$$\tilde{E}_{out}^{(2)} = \beta e^{i\Delta\phi_0} \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \cdot \frac{1}{2}$$

$$= \beta e^{i\Delta\phi_0} e^{-i\pi/4} \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{1}{\sqrt{2}}$$

is a left-circularly polarized

If we start from $\begin{pmatrix} \alpha \\ \beta e^{i\Delta\phi_0} \end{pmatrix}$ and $M_{LL} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

then $\tilde{E}_{out}^{(1)} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We use $\lambda/4$ at -45°:

$$M_{\lambda/4}(-45^\circ) = \frac{1}{2}\begin{pmatrix} 1+i & i-1 \\ i-1 & 1+i \end{pmatrix} \Rightarrow \tilde{E}_{out}^{(2)} \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

8-(3) First use a wave-plate with arbitrarily adjustable phase

$$M_{wp}(\Delta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta} \end{pmatrix}$$

and choose $\Delta = -\Delta\Phi_0$, then

$$\tilde{E}_{out}^{(1)} = M_{wp}(\Delta = -\Delta\Phi_0)\tilde{E}_{inc}$$

$$= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \tan\theta = \beta/\alpha$$

Then use a half-wave plate to rotate the linear polarized $\tilde{E}_{out}^{(1)}$ into

$$\tilde{E}_{out}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \left(M_{\lambda/2} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & +\sin\theta \end{pmatrix} \right)$$

Followed by a $\lambda/4$-plate at $+45°$

$$M_{\lambda/4}(+45°) = \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \frac{1}{2}$$

$$\Rightarrow \tilde{E}_{out}^{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \text{ no loss of power.}$$