Week #1 : Chapter 21

- Composition of Matters

$$\text{Matter} \Rightarrow \underset{(\text{Molecules})}{\overset{10^{-10}\,m}{\overset{\shortparallel}{\text{Atoms}}}} \begin{cases} \nearrow \overset{10^{-15}\,m \cdot A^{1/3},\ 1.67\times10^{-27}\,kg\ (m_p)}{\overset{\shortparallel}{\text{Protons}}}\ (Z:\ \text{atomic number}) \\ \Rightarrow \text{Electrons}\ (Z:\ \text{typically}) \\ \searrow \underset{1.67\times10^{-27}\,kg\ (m_n)}{\text{Neutron}}\ (A-Z;\ A:\ \text{atomic weight}) \end{cases}$$

Proton (positively charged): $1e = 1.602\times10^{-19}$ Coulomb (C)

Electron (negative charged): $q_e = -e = -1.602\times10^{-19}\,C$

$m_e = 9.1\times10^{-31}\,kg$

$r_e = 2.8\times10^{-15}\,m$

Example: $A(Cu) = 63.6 \Rightarrow 6.02\times10^{23}$ Cu atoms weigh 63.6 gm

$Z(Cu) = 29$

$\Downarrow\ \rho(Cu) = 8.9\ gm/cm^3$

$$\#\ \text{Cu atoms}/cm^3 = \frac{6.02\times10^{23}\ \text{atoms}}{(63.6\,gm/\rho(Cu))} \sim 9\times10^{22}/cm^3$$

Total electron charge in $1\,cm^3$ Cu

$$Q_e = Z \cdot q_e \cdot (\#\ \text{Cu atoms}/cm^3)$$

$$= -1.6\times10^{-19}(C)\times 29 \times 9\times10^{22}$$

$$\sim 4\times10^{5}\,C_{*}$$

Big ideas of E&M: Where are we heading?

1. Electric charges (moving or not) produce electric force or force field that are felt by electric charges (moving or not)

2. Moving electric charges produce magnetic force or force field that are only felt by moving electric charges

3. Time-varying magnetic force field produces electric force field that are felt by electric charges (moving or not)

4. Time-varying electric force field produces magnetic force field that are felt by moving electric charges.

5. Mutually generating E & M fields propagate as a wave

- Electrification: charge transfer ($10^{-8} \sim 10^{-7}$ C)

  $1\,nC = 10^{-9}\,C$; $1\,\mu C = 10^{-6}\,C$

  Conductors: at least one type of charge carriers is free to move about

  Insulators: neither type of charge carriers is free to move about

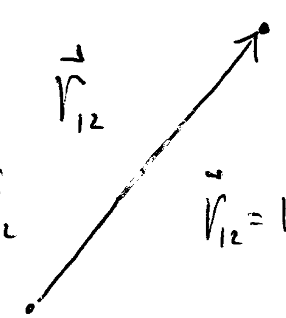
  Electrification: charge transfer between insulators as a result of forced contact and "quick" separation.

  $\boxed{\text{Electrons are being exchanged}}$

Like charges repel each other

Opposite charges attract each other

- Coulomb's law (between point charges):

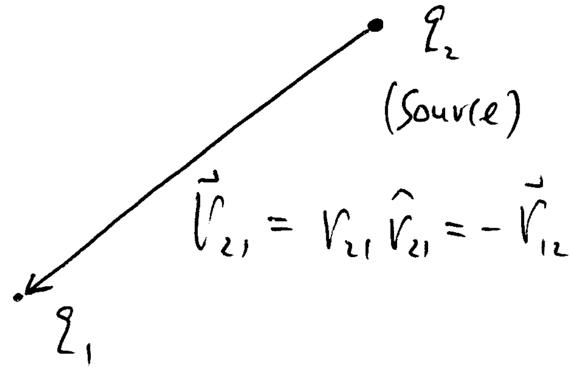


$$\vec{F}_{1 \text{ on } 2} = k \frac{q_1 \cdot q_2}{r_{12}^2} \hat{r}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 \cdot q_2}{r_{12}^2} \hat{r}_{12}$$

$$\vec{r}_{12} = r_{12} \cdot \hat{r}_{12}$$

$\epsilon_0 = 8.85 \times 10^{-12}\ C^2/N \cdot m^2$, $K = 9.0 \times 10^{9}\ N \cdot m^2/C^2$ q_1 (source)

$$\vec{F}_{2\,on\,1} = \frac{1}{4\pi\epsilon_0}\frac{q_2 \cdot q_1}{r_{21}^2}\hat{r}_{21} = -\vec{F}_{1\,on\,2}$$



$q_1 \cdot q_2$: algebraic product (with signs)

Like charges : $q_1 q_2 > 0$

Opposite charges : $q_1 q_2 < 0$

"Point charges": Distributions of q_1 & q_2 respectively $\ll r_{12}$

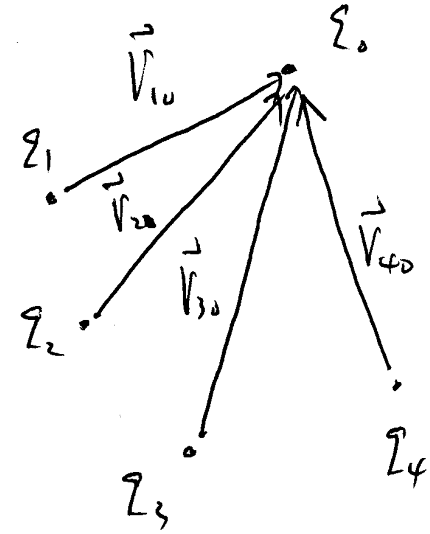
$\boxed{\text{Example 22-2}}$;

$$\left(\frac{F_e}{F_g} \sim 1 \text{ if } \frac{q}{m} \simeq 10^{-10}\ C/kg\right)^*$$

- Principle of Superposition: (vector sum)

$$\vec{F}_{on\,q_0} = \vec{F}_{1\,on\,0} + \vec{F}_{2\,on\,0} + \cdots$$

$$= \frac{q_0}{4\pi\epsilon_0}\left\{\frac{q_1}{r_{10}^2}\hat{r}_{10} + \frac{q_2}{r_{20}^2}\hat{r}_{20} + \cdots\right\}$$



$\boxed{\text{Example 22-4}}$

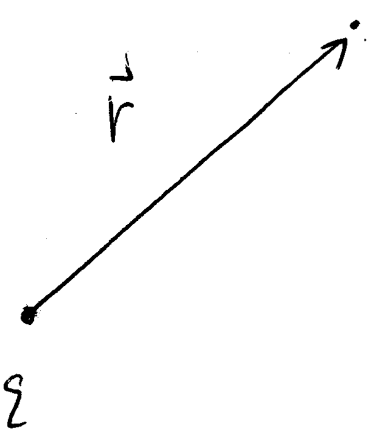
- Electric field (force field like gravitational field)

Like the mass of the Earth sets up a gravitational force field such that any massive object can feel the pull of this force field, an electric charge q sets up an electric field such that another point charge Q can feel the force exerted by q

$$F_{q\,\text{on}\,Q} = \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q}{r^2}\,\hat{r}\cdot Q \equiv \vec{E}_q \cdot Q$$

$$\vec{E}_q(\vec{r}) \equiv \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r} \qquad (\text{SI}: \ N/C)$$

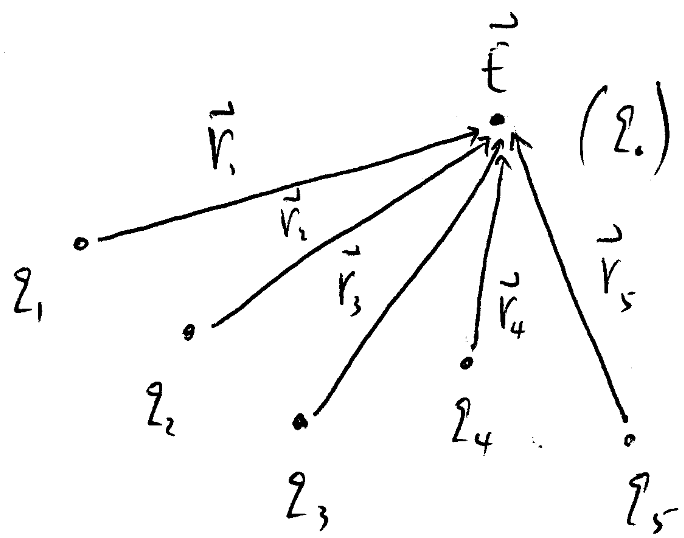
$\vec{E}_q(\vec{r})$: electric field produced by a point charge q at a field position $\vec{r}$ from q.



Force by q on a point charge Q placed at $\vec{r}$

$$\vec{F} = Q\,\vec{E}_q(\vec{r}) = Q\vec{E} \quad (\text{for simplicity})$$

Electric field produced by a collection of point charges (principle of superposition)



$$\vec{E} = \frac{\vec{F}_{\text{on } q_0}}{q_0} = \vec{E}_{q_1}(\vec{r}_1) + \vec{E}_{q_2}(\vec{r}_2) + \cdots \quad \text{(Vector sum)}$$

$$\vec{F}_{\text{on } q_0} = q_0 \vec{E} \quad (\text{as long as } \vec{E} \text{ is known})$$

Determination of $\vec{F}_{\text{on } q_0}$ is a matter of determination of $\vec{E}$ ⁂

$\boxed{\text{Example 22-9}}$

- Electric fields from continuous charge distribution

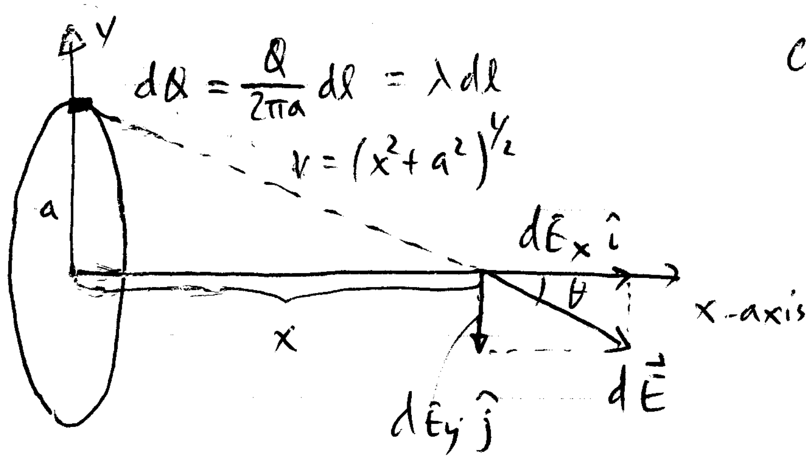
Line charge density (C/m): $\lambda = \Delta Q / \Delta L$ *

Surface charge density (C/m²): $\sigma = \dfrac{\Delta Q}{\Delta A}$ *

Volume charge density (C/m³): $\rho = \Delta Q / \Delta V$ *

* ΔL, $\sqrt{\Delta A}$, $\sqrt[3]{\Delta V}$ small compared to source-field distance

$\boxed{\text{Example 22-10}}$



$\cos\theta = \dfrac{x}{r} = \dfrac{x}{(x^2+a^2)^{1/2}}$

dE_y cancels the contribution from dl on the opposite side

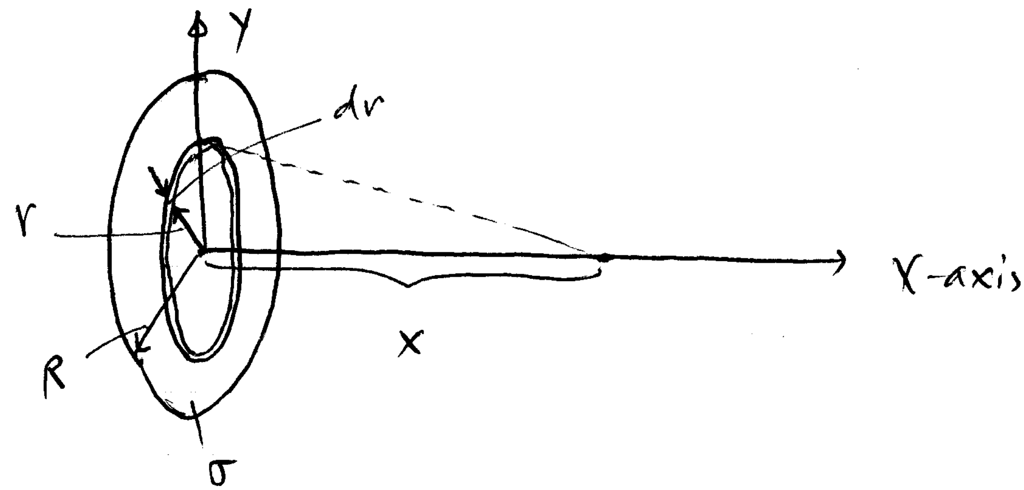
dE_x adds up: $E_x = \displaystyle\int_0^{2\pi a} dE_x = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi a} \frac{(Q/2\pi a)\,dl}{x^2+a^2}\cdot\frac{x}{(x^2+a^2)^{1/2}}$

$= \dfrac{1}{4\pi\epsilon_0}\dfrac{Q\cdot x}{(x^2+a^2)^{3/2}} \Rightarrow \vec{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Qx}{(x^2+a^2)^{3/2}}\hat{i}$

$x \gg a$: "point charge": $E_x = \dfrac{1}{4\pi\epsilon_0}\cdot\dfrac{Q}{x^2}$

$\boxed{\text{Example 22-11 : read yourself}}$

$\boxed{\text{Example 22-12}}$:



From the thin ring of radius r and width dr

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{x}{(x^2+r^2)^{3/2}}\cdot(2\pi r\,dr\,\sigma)$$

$$E_x = \int_0^R dE_x = \frac{x\sigma}{4\pi\epsilon_0}\int_0^R \frac{2\pi r\,dr}{(x^2+r^2)^{3/2}}$$

$$= \frac{\sigma}{2\epsilon_0}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right]$$

$$\vec{E} = E_x\,\hat{i} = \frac{\sigma}{2\epsilon_0}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right]\hat{i}$$

Limiting cases:

Point charge ($R \ll x$): $\vec{E} \cong \frac{\sigma}{2\epsilon_0} \cdot \frac{R^2}{2x^2}\hat{i} = \frac{1}{4\pi\epsilon_0}\frac{\pi R^2\sigma}{x^2}\hat{i}$

Infinitely large plane of charge ($R \gg x$): $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{i}$

How large a plane does one need? $\vec{E} = (0.9)\frac{\sigma}{2\epsilon_0}\hat{i}$ for $\frac{x}{R} = 0.1$

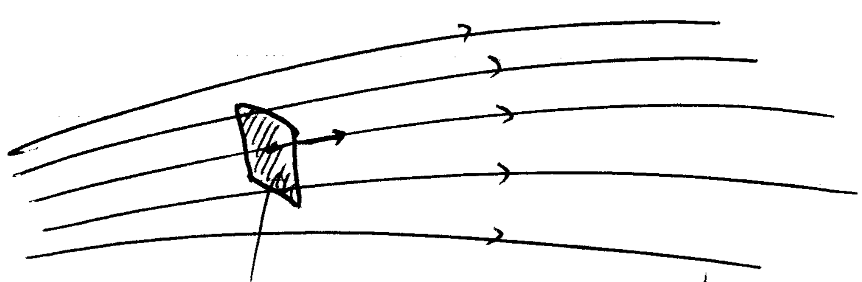
✱

- Electric Field Lines:

A set of directed lines representing (depicting) electric field

Direction: along $\hat{E}$ locally

Number density: equal to magnitude of $|\vec{E}|$



ΔA perpendicular to $\vec{E}$

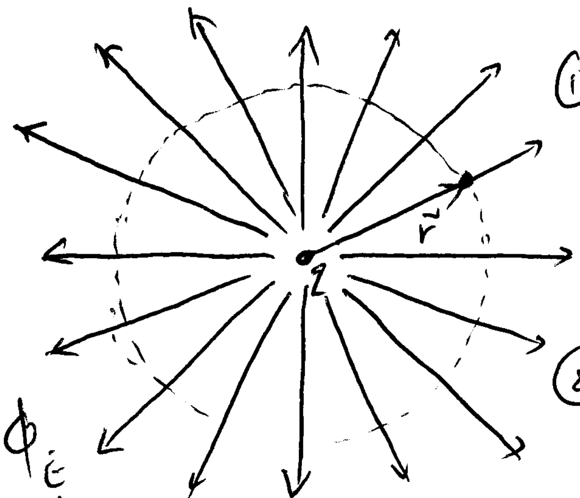
Φ_E: $N \cdot m^2/C$

electric flux

$$|\vec{E}| = \frac{\text{Number of field lines passing through } \Delta A}{\Delta A} = \frac{\Delta\Phi_E}{\Delta A}$$

Electric field lines from a point charge

Draw $\Phi_{\vec{E}} = |q|/\epsilon_0$ lines symmetrically outward for a positive charge q.

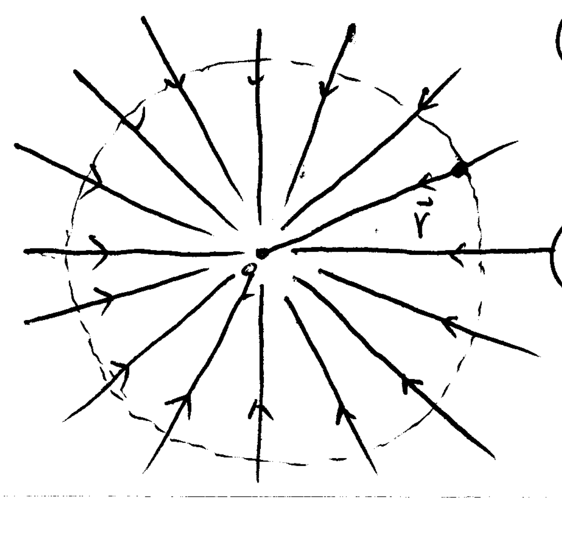


$\Phi_{\vec{E}}$
(out)

① Direction of the lines is along radial direction $\hat{r}$;

② Density of the field lines $= \dfrac{\Phi_{\vec{E}}}{4\pi r^2} = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$
$= |\vec{E}|$

Draw $\Phi_{\vec{E}} = |q|/\epsilon_0$ lines symmetrically inward for a negative charge $q < 0$.



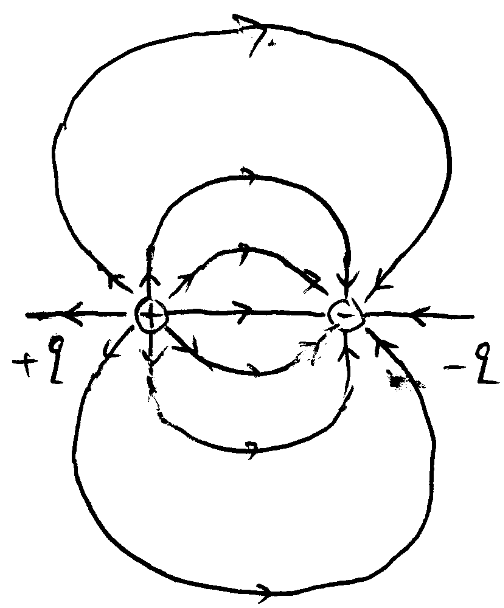
$\Phi_{\vec{E}}$
(in)

① Direction of the lines along the opposite of the radial direction: $-\hat{r}$

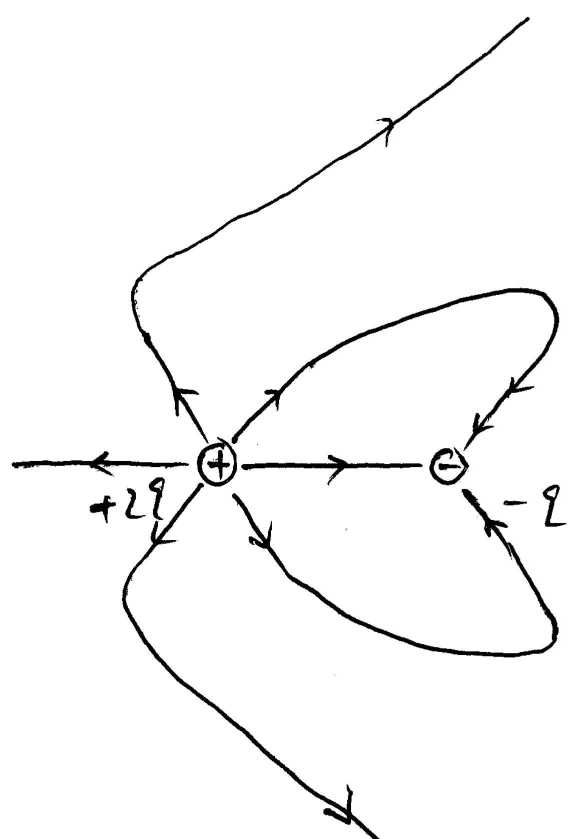
② Number density

$$\frac{\Phi_{\vec{E}}}{4\pi\epsilon_0} = \frac{\Phi_{\vec{E}}}{4\pi r^2} = \frac{1}{4\pi\epsilon_0}\frac{|q|}{r^2} = |\vec{E}|$$

Electric field lines from two point charges



To a far field point, there appears no net "point" charge in this region. ⇒ Electric field lines must thin out to zero



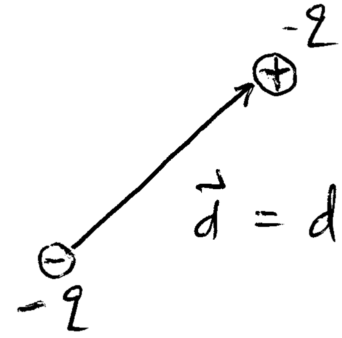
To a faraway field point, there appears a net positive point charge q in this region, and thus "three" net field lines "escape", eventually symmetrically "radiate" outward.

- Motion and behaviors of electric charges in electric fields

$\boxed{\text{Example 22-7, read it yourself, similar to a point mass near the surface of Earth}}$

- electric dipole in electric fields

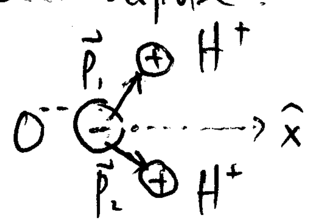
An electric dipole is a pair of charges of equal magnitude but with opposite sign, with the positive charge separated from the negative charge by a position vector $\vec{d}$



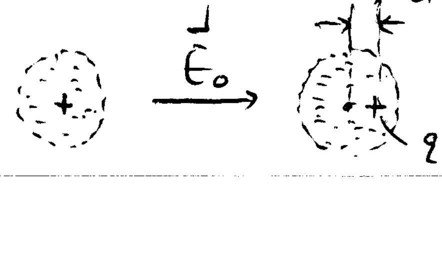
$\vec{d} = d\,\hat{d}$

Dipole moment $\vec{p} \equiv q\vec{d} = qd\,\hat{d}$

Permanent dipole: water molecules

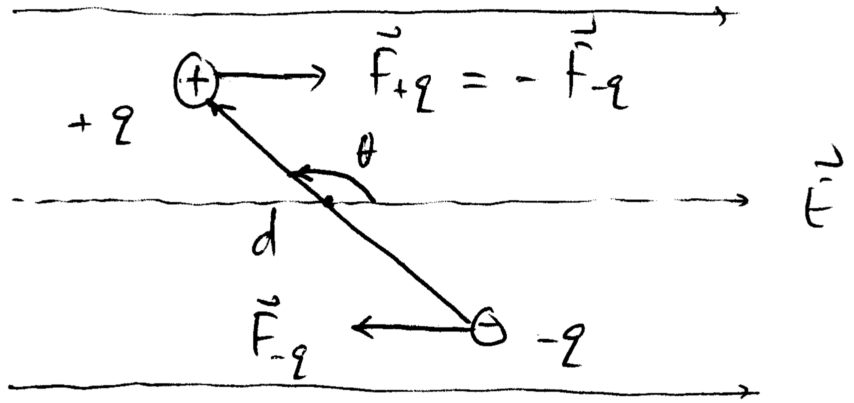


$\vec{p} = \vec{p}_1 + \vec{p}_2 = 6.2\times10^{-30}\,C\cdot m\,\hat{i}$

Induced electric dipole: 

$\vec{p} = q\vec{d} \propto \vec{E}_0$

Electric dipole (permanent or induced) in a uniform $\vec{E}$



* Field line demonstration
$\vec{p}$ prefers to align along $\vec{E}$ *

① Net force on the dipole is zero. $\vec{F}_{+q} = -\vec{F}_{-q} = q\vec{E}$

② But $\vec{F}_{+q}$ and $\vec{F}_{-q}$ are along the two lines that are apart by $d\sin\theta$, thus exerting a torque

$$|\vec{\tau}| = qEd\sin\theta = (qd\sin\theta)\cdot E = |\vec{p}\times\vec{E}|$$

$$\vec{\tau} = \vec{p}\times\vec{E}$$ (causing clockwise rotation in this case)

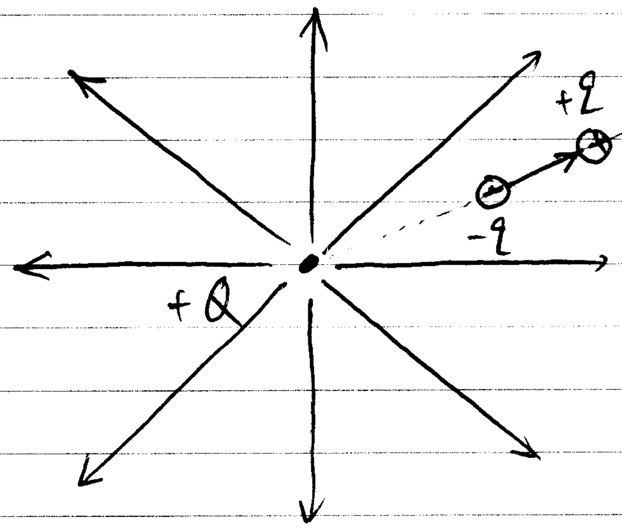
③ Potential energy $U(\theta)$ of $\vec{p}$ in $\vec{E}$:
Potential energy U is gained by rotating $\vec{p}$ counterclockwise against electric torque $\vec{\tau}$:

$$U(\theta) - U(0) = \int_0^\theta pE\sin\theta \cdot d\theta = -pE\cos\theta + pE$$

$$\therefore\ U(\theta) = -pE\cos\theta = -\vec{p}\cdot\vec{E}$$ *

- Electric dipole $\vec{p}$ in a non-uniform

(Attraction of water by charged rod/plastic)



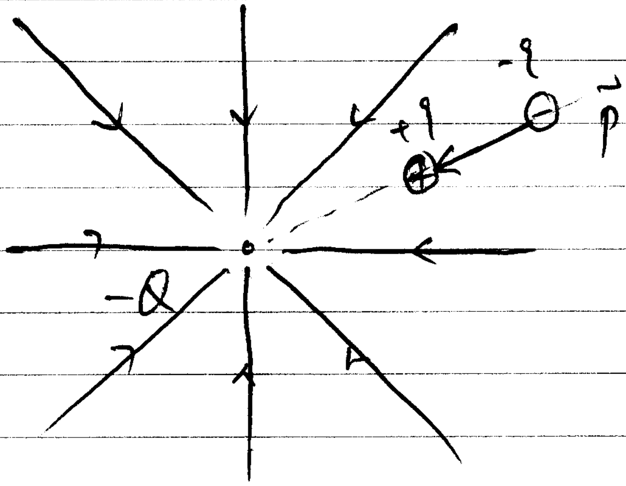
$\vec{p}$ first aligns along the direction of the mean electric field

$$\langle \vec{E} \rangle = \frac{1}{2}\left(\vec{E}_{-q} + \vec{E}_{+q}\right)$$

Since $-q$ is closer to $+Q$ than $+q$, $|\vec{E}_{-q}| > |\vec{E}_{+q}|$.

Thus, $\vec{p}$ is attracted to Q (positive)

This is true for permanent/induced dipole moment.



Since $+q$ is closer to $-Q$ than $-q$.

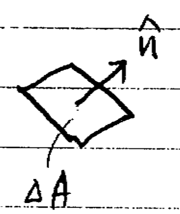
$|\vec{E}_{+q}| > |\vec{E}_{-q}|$.

Thus $\vec{p}$ is attracted to $-Q$, source of the non-uniform $\vec{E}$ field.

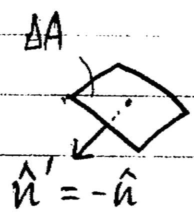
Week #2. Chapter 22 (Gauss' law)

- Electric flux: electric field lines through a directed area

Directed area $\Delta\vec{A}$ (Vector area): area of a flat surface ΔA multiplied by a chosen unit vector $\hat{n}$ along its normal

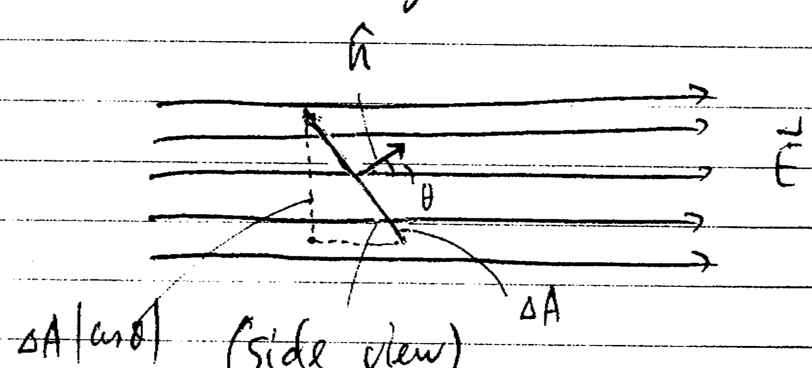


$\Delta\vec{A} = \Delta A\,\hat{n}$



$\Delta\vec{A}' = \Delta A\,\hat{n}' = -\Delta A\,\hat{n}$ (two choices)

Electric flux through a directed area $\Delta\vec{A}$:



$|\Delta\Phi_E| = |\vec{E}|\cdot\Delta A\cdot|\cos\theta|$

Define:

$\Delta\Phi_E = \vec{E}\cdot\Delta\vec{A}$

$\Delta\phi_E > 0$: $\hat{E}\cdot\hat{n} > 0$, electric field lines leaving the directed area $\Delta\vec{A}$

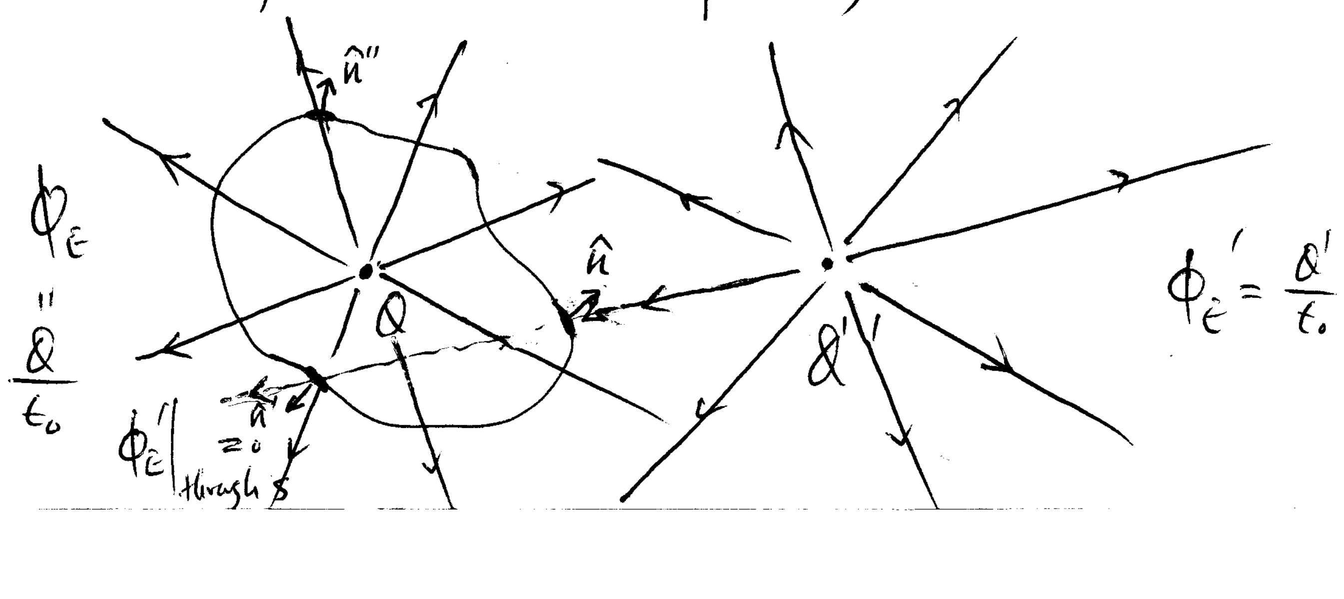
$\Delta\phi_E < 0$: $\vec{E}\cdot\hat{n} < 0$, electric field lines entering the directed area $\Delta\vec{A}$

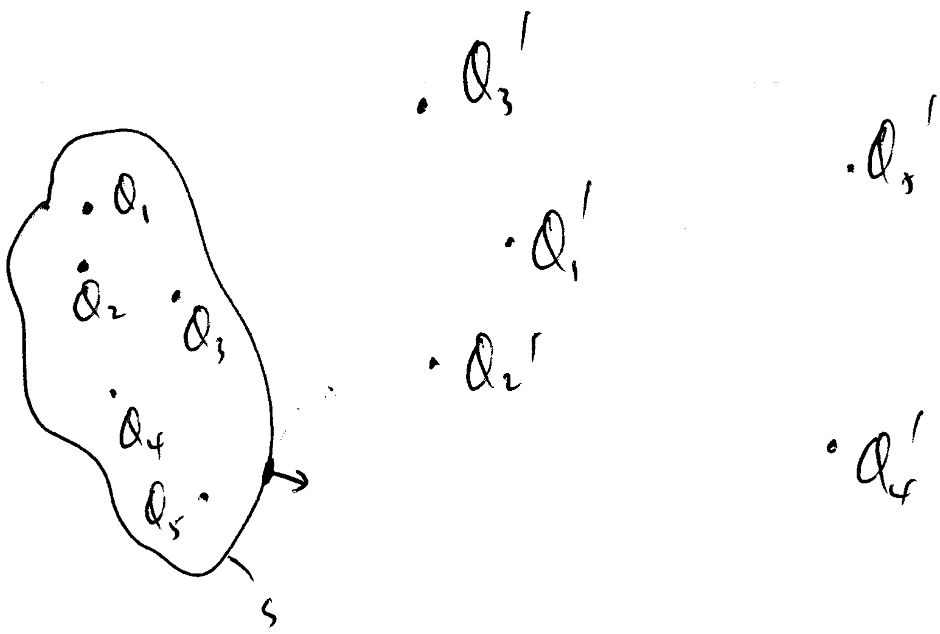
- Gauss' law:

Electric field lines leaving a closed surface S equal the total charge inside S divided by ϵ_0

$$\oiint_S \vec{E}\cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0}$$

The unit vector of each directed area element $d\vec{A}$ is always chosen to be pointing outward.





$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{A} = \oint\!\!\!\oint_S \left(\vec{E}_{Q_1} + \vec{E}_{Q_2} + \cdots + \vec{E}_{Q_5} + \vec{E}_{Q_1'} + \vec{E}_{Q_2'} + \cdots + \vec{E}_{Q_5'}\right)\cdot d\vec{A}$$

$$= \oint\!\!\!\oint_S \vec{E}_{Q_1}\cdot d\vec{A} + \oint\!\!\!\oint_S \vec{E}_{Q_2}\cdot d\vec{A} + \cdots + \oint\!\!\!\oint_S \vec{E}_{Q_5}\cdot d\vec{A}$$

$$+ \oint\!\!\!\oint_S \vec{E}_{Q_1'}\cdot d\vec{A} + \oint\!\!\!\oint_S \vec{E}_{Q_2'}\cdot d\vec{A} + \cdots + \oint\!\!\!\oint_S \vec{E}_{Q_5'}\cdot d\vec{A}$$

$$= \frac{Q_1}{\epsilon_0} + \frac{Q_2}{\epsilon_0} + \cdots + \frac{Q_5}{\epsilon_0}$$

$$= \frac{(Q_1 + Q_2 + \cdots + Q_5)}{\epsilon_0}$$

$$= \frac{Q_{inside}}{\epsilon_0}$$

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{inside}}{\epsilon_0} \quad \text{(Gauss' law)}$$

It doesn't mean at all that $\vec{E}$ is only produced by Q_{inside}!!

- Application of Gauss' law (What's the use of it??)
  — Finding $\vec{E}$ under symmetric or limiting situations

Example 23-5

Example 23-6, read yourself, and compare with example 22-11 in the limit that the line is long*

Example 23-7, may read yourself, compare with Example 22-12

- Charges on conductor surfaces and electric fields at conductor surfaces

Under static conditions (nothing moves no more), the electric field inside a conductor is always zero (otherwise freely movable electric charge carriers would move under the electric field until the latter is perfectly cancelled out): $\vec{E}_{inside} = 0$

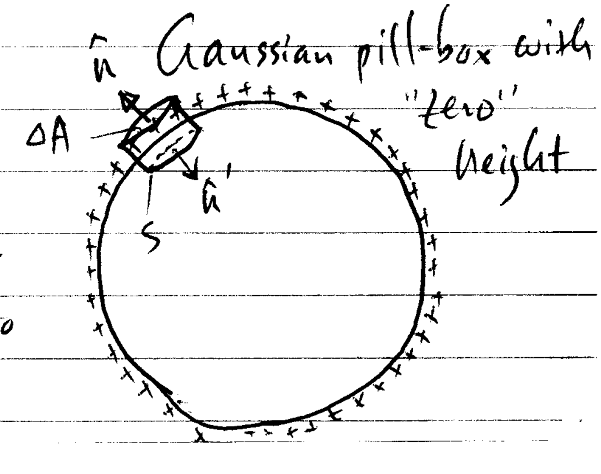
No net (unbalanced) electric charges can remain inside a conductor.
(By enclosing any volume inside a conductor, since

$$\oint_S \vec{E}\cdot d\vec{A} = \oint_S \vec{E}_{inside}\cdot d\vec{A} = 0 = \frac{Q_{inside\ S}}{\epsilon_0}$$

net charges inside any volume in a conductor is zero.)

Net or unbalanced electric charges can only reside on the surface of a conductor.

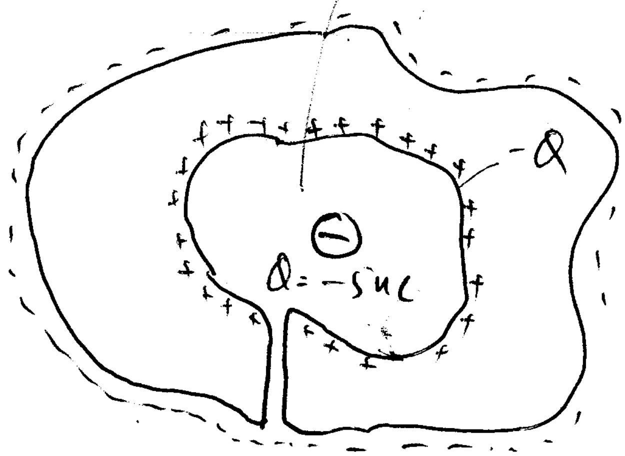
Electric field outside a conductor surface:



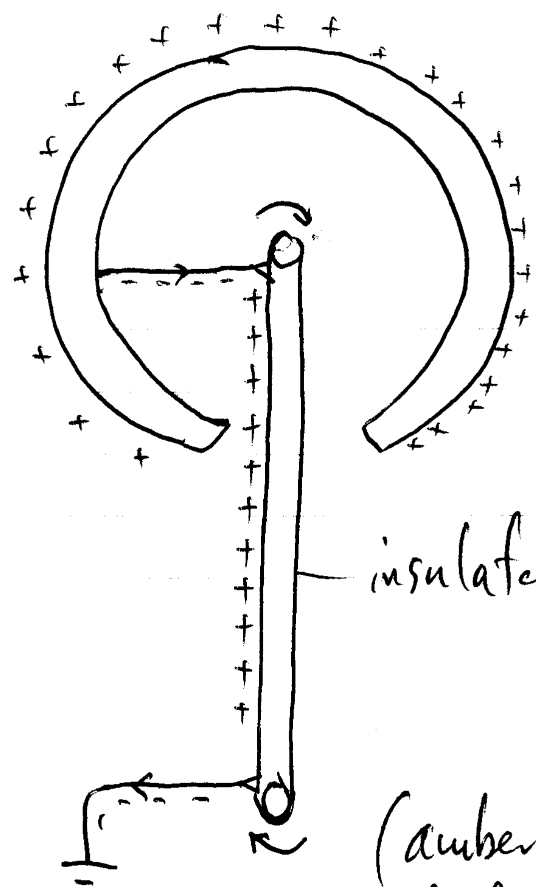
$$\oint_S \vec{E}\cdot d\vec{A} = \vec{E}_{outside}\cdot\hat{n}\cdot\Delta A = \sigma\cdot\Delta A/\epsilon_0$$

$$\vec{E}_{outside}\cdot\hat{n} = \sigma/\epsilon_0 \quad *$$

Example 23-10



Van der Graaf:



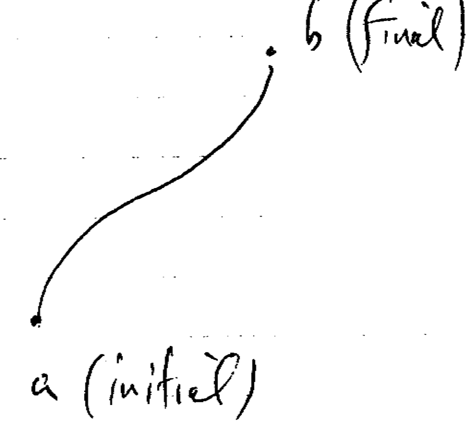
Week #3 : Chapter 23

- Electric potential energy U of charges in an electric field: "potential energy originated from electro-static forces".

Like a massive object in a gravitational field (a coke can at the surface of the Earth), a charged object in an electric field experiences a force

$$\vec{F}_E = q\vec{E} \quad \text{(electro-static)}$$

Alone, $\vec{F}_E$ accelerates the charged (q) object and does work when q is moved from a to b, gaining kinetic energy K



Work done by $\vec{F}_E$ on q

= Gain of kinetic energy of a q-charged object,

$K_b - K_a$

$$\boxed{W(\vec{F}_E \text{ on } q) = K_b - K_a}$$

(SI unit. Joules)

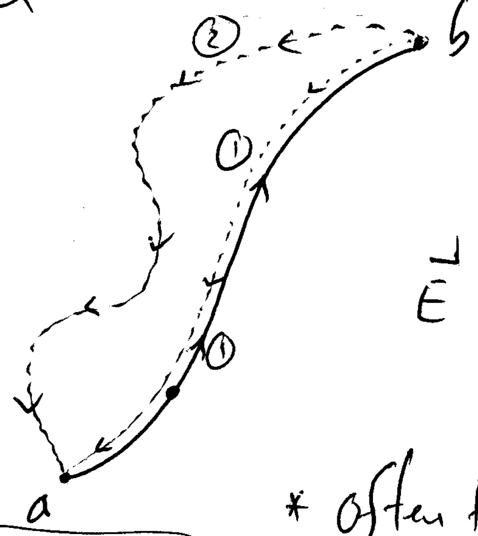
$\vec{F}_E$ alone

$\vec{F}_E$ is "conservative" (like gravitational forces):

1) Work done against $\vec{F}_E$ as a charged object is moved from b to a, by an external force or at the expense of the kinetic energy of the object (K_b) is not "wasted"

2) The equal amount of work is returned as $\vec{F}_E$ works to move the object from a back to b in form of the kinetic energy of the object or the work done to an external system (e.g., heating up a light bulb, moving a mechanical motor).

3) The work done by $\vec{F}_E$, or against $\vec{F}_E$, is independent of the route that the charged object takes to get from an initial position (e.g., a) to a final position (e.g., b).



A charged object q in $\vec{E}$ has a potential energy U_q*

$$\boxed{U_q(a) - U_q(b) = \text{Work done by } \vec{F}_E = q\vec{E} = \int_a^b \vec{F}_E \cdot d\vec{l} = q\int_a^b \vec{E}\cdot d\vec{l}}$$

* often that is all we really care !!!

Equivalently, the electric potential energy change is equal to the work done against $\vec{F}_E$ when q is moved from an initial position a to a final position b:

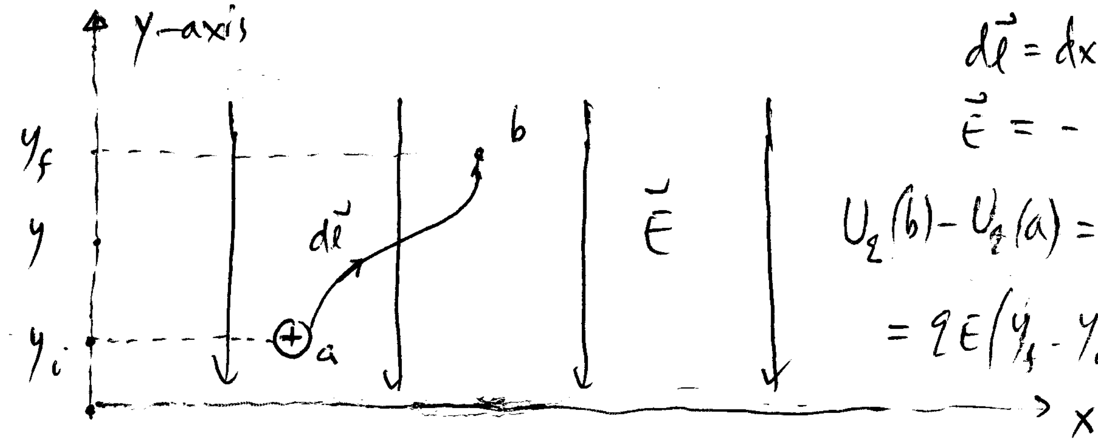
$$U_q(b) - U_q(a) = \int_a^b \vec{F}_{ext} \cdot d\vec{l} = \int_a^b (-\vec{F}_E) \cdot d\vec{l}$$

$$= -q \int_a^b \vec{E} \cdot d\vec{l}$$

$$\therefore \boxed{U_q(b) - U_q(a) = -q \int_a^b \vec{E} \cdot d\vec{l} = q \cdot \left(-\int_a^b \vec{E} \cdot d\vec{l} \right)}$$

Example: Potential energy a charged particle in a uniform electric field $\vec{E}$

$$\boxed{U_q(y) = qE \cdot y}$$



$d\vec{l} = dx\,\hat{i} + dy\,\hat{j}$

$\vec{E} = -E\,\hat{j}$

$$U_q(b) - U_q(a) = q \int_a^b (-\vec{E}) \cdot d\vec{l}$$

$$= qE(y_f - y_i)$$

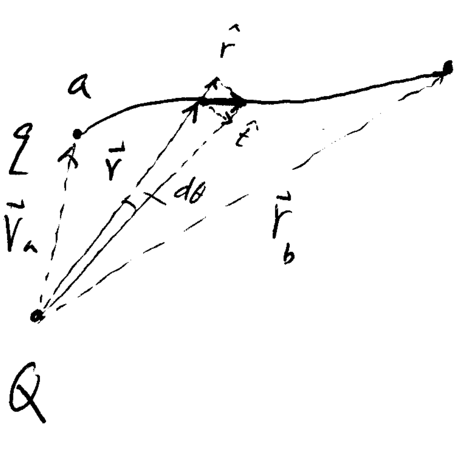
Unlike massive objects ($m>0$) in a gravitational field, in a same electric field $\vec{E}$, whether $U_q(b)$ is higher or lower than $U_q(a)$ also depends on whether $q>0$ or $q<0$.

① $q>0$: U_q increases in the direction opposite to $\vec{E}$

② $q<0$: U_q decreases in the direction opposite to $\vec{E}$

(unlike gravity !!)

<u>Example</u>: Electric potential energy of a point charge q in the electric field produced by another point charge Q



$$U_q(b) - U_q(a) = q\int_a^b (-\vec{E})\cdot d\vec{l}$$

$$= -q\cdot\int_a^b \frac{K\cdot Q}{r^2}\,\hat{r}\cdot(\hat{r}\,dr + r\,d\theta\,\hat{t})$$

$$= -KqQ\cdot\int_a^b \frac{dr}{r^2} = KqQ\int_{r_a}^{r_b} d\left(\frac{1}{r}\right)$$

$$d\vec{l} = dr\,\hat{r} + (r\,d\theta)\cdot\hat{t}$$

$$= \frac{KqQ}{r_b} - \frac{KqQ}{r_a}$$ (independent of how q reaches b from a)

$$U_q(r) = \frac{kq \cdot Q}{r} + \text{constant}$$

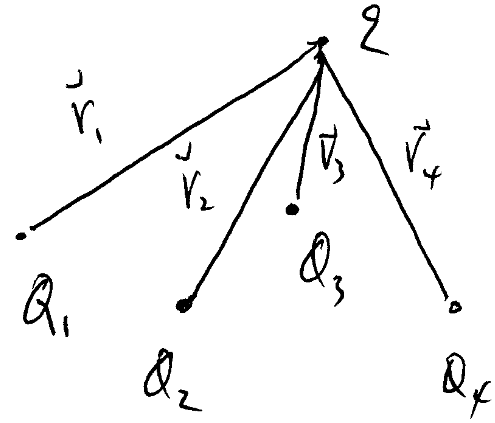
Taking $U_q(\infty) = 0$, constant is set to zero

$$\boxed{U_q(r) - U_q(\infty) = \frac{kqQ}{r}}$$

<u>Principle of superposition</u>:

electric potential energy of a point charge q in the electric field produced by a set of point charges (Q_1, Q_2, ...)

$$U_q(r) - U_q(\infty) = \sum_{n=1} \frac{kqQ_n}{r_n}$$



For continuous charge distributions,

$$U_q - U_q(\infty) = \int \frac{kq\,dQ}{r}$$

(volume, surface, line)

- Electric potential energy of a collection of point charges in a given configuration (arrangement).

Potential energy of these charges in a given configuration is the work done AGAINST the electro-static forces between these charges, when they are brought from being infinitely apart to the configuration.

Two charges (q_1, q_2, separated by r_{12})

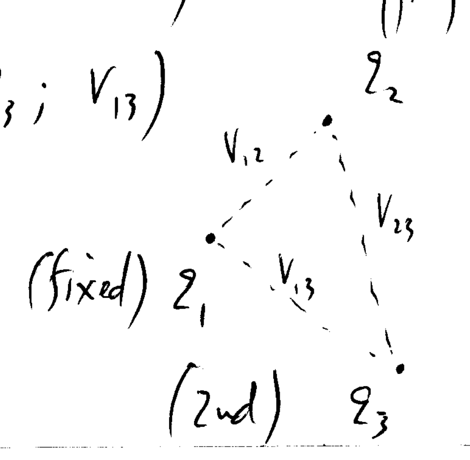
$$U(q_1, q_2; r_{12}) = \frac{k q_1 \cdot q_2}{r_{12}}$$

Three charges (q_1, q_2, q_3; r_{12}, r_{23}, r_{32})

$$U(q_1, q_2, q_3; r_{12}, r_{21}, r_{23}) = U(q_1, q_2; r_{12})$$

$$+ U(q_2, q_3; r_{23}) + U(q_1, q_3; r_{13})$$

$$= \frac{k q_1 q_2}{r_{12}} + \frac{k q_2 \cdot q_3}{r_{23}} + \frac{k q_1 \cdot q_3}{r_{13}}$$



- Electric potential V in an electric field $\vec{E}$

Electric potential V is a scalar function of spatial coordinate. The difference of the electric potential between two points in space, a and b, equals to the work done on a unit positive charge against the electrostatic force produced by $\vec{E}$ on the charge as it is moved from a to b:

$$\boxed{V_b - V_a \equiv -\int_a^b \vec{E}\cdot d\vec{l}}$$

(SI units: volt or V
$1V = 1 J/C = 1 N\cdot m/C$)

- Potential energy difference for a charge q in the electric field is

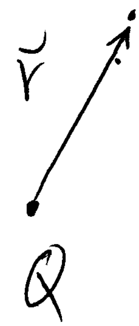
$$\boxed{U_q(b) - U_q(a) = q\left(V_b - V_a\right)} \Rightarrow \boxed{V_b - V_a = \frac{U_q(b) - U_q(a)}{q}}$$

- Electric field $\vec{E}$ and the gradient of $V(\vec{r})$: $\boxed{\vec{E} = -\nabla V}$

$$V(\vec{r}+d\vec{r}) - V(\vec{r}) = \underbrace{\left(\frac{\partial V}{\partial x}\right)_{\vec{r}}\cdot dx + \left(\frac{\partial V}{\partial y}\right)_{\vec{r}}\cdot dy + \left(\frac{\partial V}{\partial z}\right)_{\vec{r}}}_{(\nabla V)_{\vec{r}}\cdot d\vec{r}}\cdot dz = -\vec{E}\cdot d\vec{r}$$

- Electric potential V in the electric field of a point charge Q

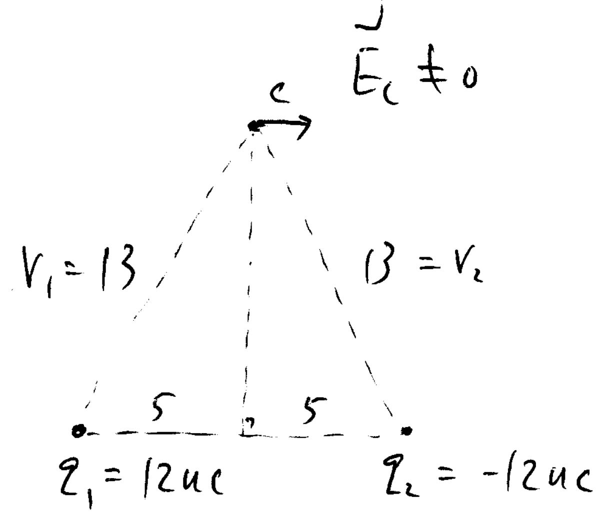
$$V(\vec{r}) - V(\infty) = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r}$$



Assuming $V(\infty) = 0$ (reference)

$$V(\vec{r}) = \frac{kQ}{r}$$

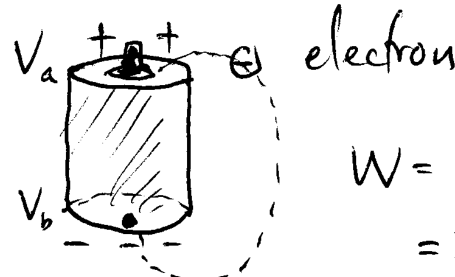
Example 24-4:



$$V_c = \frac{kq_1}{r_1} + \frac{kq_2}{r_2}$$

$$= \frac{k}{r_1}(q_1 + q_2) = 0$$

Example: Moving an electron



Work done $= q_e(V_b - V_a)$

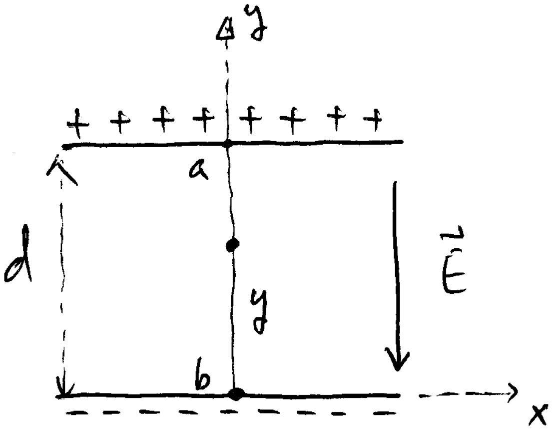
$= (-e)(-1.5\,V)$

$W = 1.5\,eV$

$= 2.4 \times 10^{-19}\,J$

✻

Example 24-9



Against $\vec{E}$, one does positive work when moving a positive charge along positive y-axis:

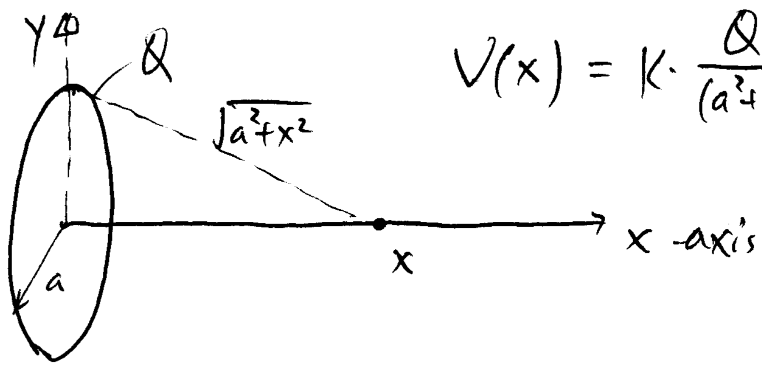
$$V(y) - V(0) = V(y) - V_b = Ey$$

Potential difference between the two plates

$$V_a - V_b = E \cdot d \qquad E = \frac{V_a - V_b}{d} \quad (V/m)$$

Electric potential increases in directions against $\vec{E}$ ※

Example 24-11.



$$V(x) = K \cdot \frac{Q}{(a^2+x^2)^{1/2}}$$

$$E_x = -\frac{\partial V}{\partial x} = K\frac{Qx}{(a^2+x^2)^{3/2}}$$

- Line integration of an electric field along a close loop is zero.

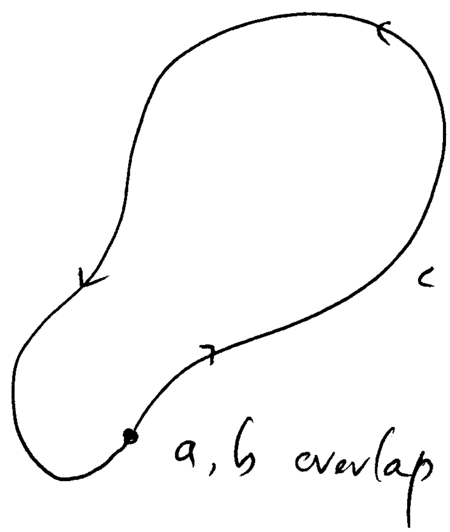
$$\oint_c \vec{E} \cdot d\vec{l} = 0 \qquad \text{(Second Maxwell's equation)}$$

Since the net work done on a charge q over a close loop by the electro-static force $\vec{F}_E = q\vec{E}$ is zero.

$$(V_b - V_a)\Big|_{a,b \text{ overlap}}$$

$$= -\oint_c \vec{E} \cdot d\vec{l}$$

$$= 0$$ (We haven't gone anywhere!)



- Equi-potential surfaces:

All points in space such that $V(\vec{r}) = \text{constant}$.

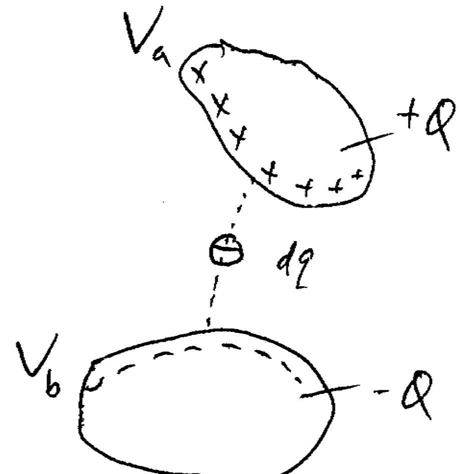
Week #4. Chapter 24

- Moving electric charges against electric forces allows storage of energy (electric potential energy)
- To retrieve stored electric potential energy with ease requires the movable electric charges to be mobile, and thus the "stored" charges must be on conductors.

- Capacitor and capacitance C of a capacitor

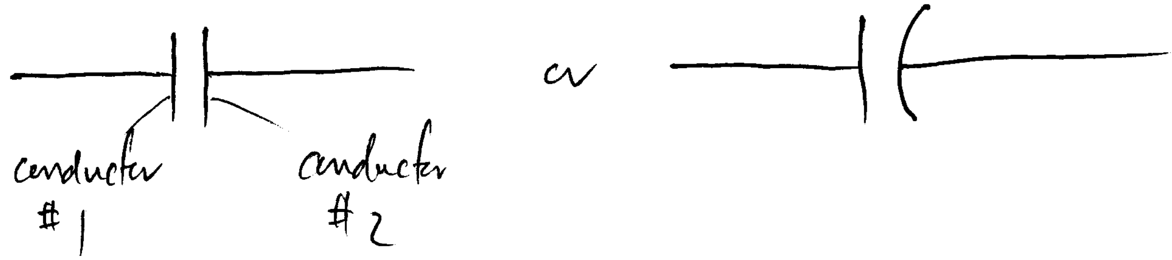
A pair of (initially neutral) conductors forms a capacitor that can store electric potential energy.

By moving a net charge Q from one conductor to another, a potential difference is established between the two conductors, $V_a - V_b$, and it is proportional to Q with the proportionality constant C only depending upon the geometry of the two conductors



$$Q = C(V_a - V_b) \qquad (\text{SI unit: Farad} = C^2 \cdot N/m)$$

C: capacitance of a capacitor (unit: Farad $= C^2/Nm$)



- stored electric potential energy in a capacitor equals the work done by moving $+Q$ from b to a

$$W = \Delta U = \int_0^Q (V_a - V_b)_q \cdot dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

$$\therefore \quad \Delta U = \frac{Q^2}{2C} = \frac{C}{2}(V_a - V_b)_Q^2$$

Equivalently, if $-Q$ is being moved (electrons), $dq > 0$

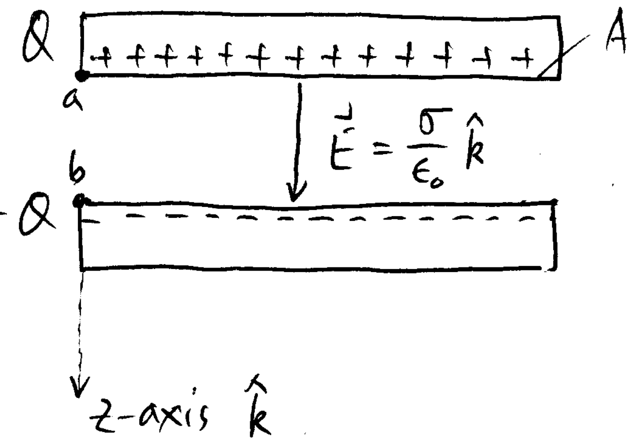
$$W = \Delta U = \int (V_b - V_a)_q (-dq) = \int (V_a - V_b)_q \cdot dq$$

$$= \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

*

• Capacitance of a parallel-plate capacitor

Two conducting plates separated by d, and having an overlapping area A.



When $d \ll \sqrt{A}$, by symmetry Q and $-Q$ are uniformly distributed. The electric field E inside is uniform and contributed by two sheets of charges

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{k} + \frac{(-\sigma)}{2\epsilon_0}(-\hat{k}) = \frac{\sigma}{\epsilon_0}\hat{k} = E\hat{k}.$$

Potential difference $V_a - V_b = Ed = \frac{\sigma}{\epsilon_0}d = \frac{Q}{\epsilon_0 A/d}$

$$\therefore C = \frac{Q}{V_a - V_b} = \frac{\epsilon_0 \cdot A}{d} \quad \text{(geometry only)}$$

C increases with A: larger overlapping area enables more charge storage with the same $V_a - V_b$.

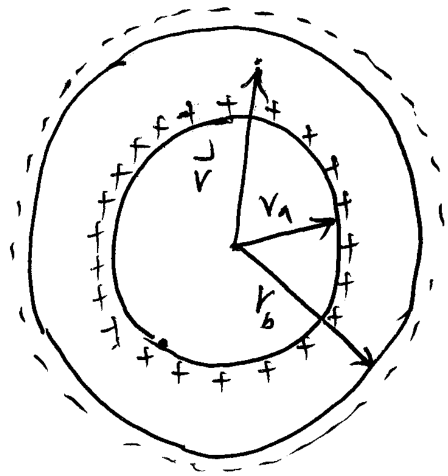
C decreases with d: larger separation requires less charge transfer to reach same $V_a - V_b$.

Example 25-2 :

Example 25-3 : Two concentric spherical conducting surfaces

From Gauss' law,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\,\hat{r}$$



Potential difference between inner and outer shell

$$V_a - V_b = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

$$C = \frac{Q}{V_a - V_b} = \frac{4\pi r_a r_b \epsilon_0}{r_b - r_a} \; *$$

* C : Farad = C^2/Nm

$\epsilon_0 = 8.85 \times 10^{-12}\, C^2/Nm^2$

$= 8.85 \times 10^{-12}\, F/m$

$= 8.85\, pF/m$

When $\Delta r \equiv r_b - r_a \ll r_a$, $4\pi r_a r_b \simeq 4\pi r_a^2$, $\Delta r \simeq d$

$$C \simeq \frac{\epsilon_0 A}{d} \text{ (parallel-plate capacitor)}$$

When $r_b = +\infty$, C (single spherical conducting sphere) $\simeq 4\pi r_a \epsilon_0$

Example 25-4 : read it yourself / Discussion

Electric field from a line charge or charges on a long cylinder

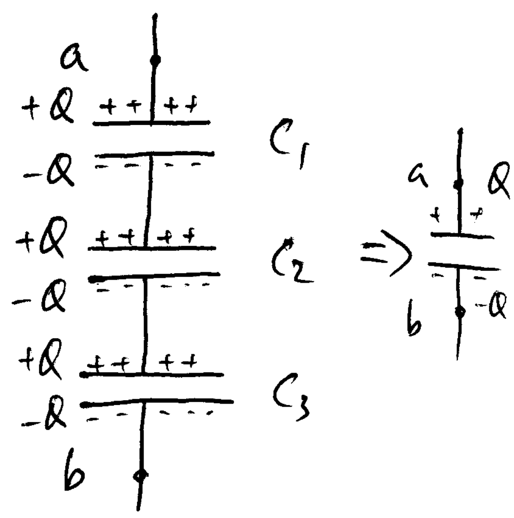
Electric potential from a line charge or charges on a long cylinder

Capacitance (BNC) of two concentric cylindrical shells of conductors per unit length ※

Capacitors in series and in parallel:

Capacitors in series:

(effectively increasing d, thus the total $C \downarrow$)



$$\frac{Q}{C} = V_a - V_b = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

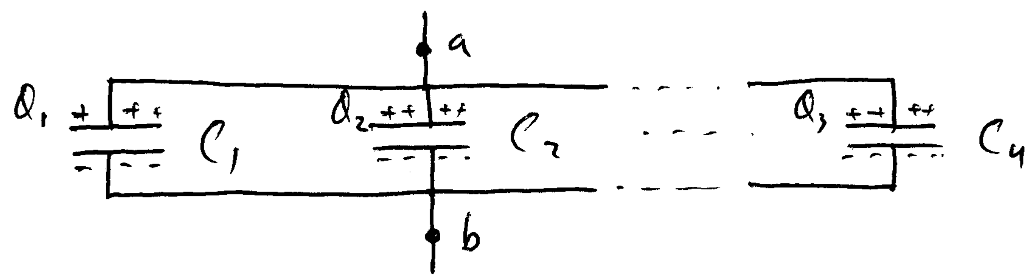
$$\therefore \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots}$$

With n identical capacitors in series, d is increased by a factor of n, and thus C is expected to drop by a factor of n:

$$1/C = 1/C_1 + \cdots + 1/C_n = n/C_1 \quad \text{or} \quad C = C_1/n.$$

Capacitors in parallel:

Effectively increasing A, thus the total $C \uparrow$.



$$\boxed{C = \frac{Q}{V_a - V_b} = \frac{Q_1 + Q_2 + \cdots + Q_n}{V_a - V_b} = C_1 + C_2 + \cdots + C_n}$$

When n identical capacitors are in parallel, d is fixed, only A is increased by a factor of n, and thus $C \sim \epsilon_0 A/d$ is expected to increase by n

$$C = C_1 + C_2 + \cdots + C_n = n C_1$$

*

Example 25-6

Example 25-7

· Dielectrics

Insulating materials stuffed inbetween the two conductors of a capacitor that make moving electric charges move easily, thus for the same charge Q, stored electrostatic energy is less by a factor K

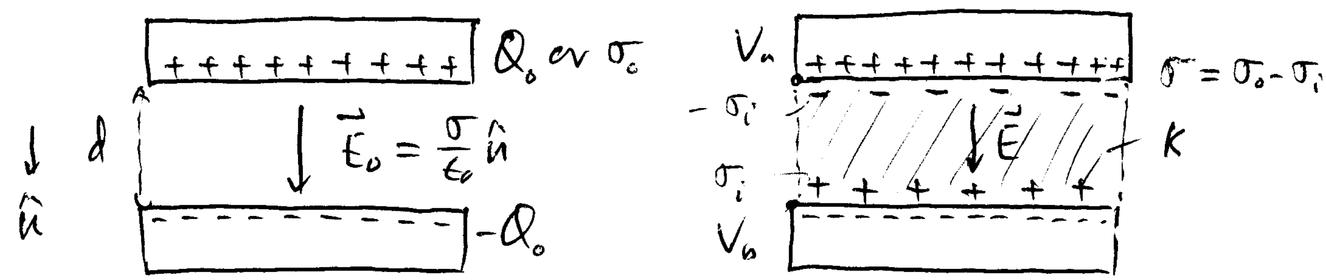
$$\Delta U_K = \frac{Q^2}{2C_K} = \frac{1}{K}\Delta U = \frac{1}{K}\left(\frac{Q^2}{2C_0}\right)$$

K : dielectric constant of an insulating material (dielectrics)

$$C_K = K C_0 \qquad K = \frac{C_K}{C_0}$$

Dielectric effect (K) comes from the tendency of a dielectric to reduce the external applied electric field by a factor of K. Such a tendence is a result of motions (restricted) of two types of electric charges along opposite directions under the applied electric field.

- Reduction of electric field in a dielectric and the capacitance of a capacitor filled with the dielectric



$$E = \frac{\sigma_0 - \sigma_i}{\epsilon_0} = \frac{1}{K} E_0 = \frac{\sigma_0}{K\epsilon_0} \qquad \text{(dropped by } K\text{)}$$

$$(V_a - V_b) = Ed = \frac{1}{K}(E_0 d) = \frac{1}{K}(V_a - V_b)\Big|_0 \qquad (\downarrow \text{ by } K)$$

$$\boxed{C = \frac{Q_0}{V_a - V_b} = \frac{Q_0}{(V_a - V_b)_0 / K} = K\frac{Q_0}{(V_a - V_b)_0} = K C_0}$$

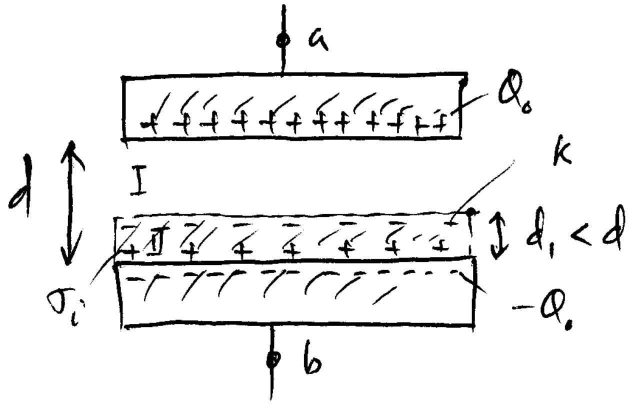
From $(\sigma_0 - \sigma_i) = \sigma_0/\epsilon_0$, $\boxed{\sigma_i = \frac{K-1}{K}\sigma_0}$

induced surface charge

Example: Water has $K = 80$, $\sigma_i \simeq \sigma_0$, so that the electric field inside water is almost totally "cancelled". Other materials are not so good. (But water is often contaminated with ions).

Example: Capacitance of a partly filled capacitor

Method #1: (from $C = Q/V$)



In Region I,

$$E_I = \frac{\sigma_0}{\epsilon_0} = \frac{Q_0}{\epsilon_0 A}$$

In region II,

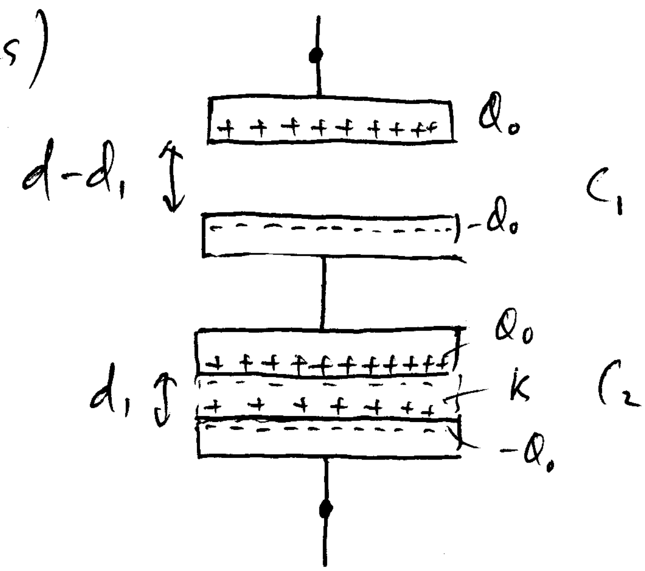
$$E_{II} = \frac{E_I}{k} = \frac{1}{k}\left(\frac{Q_0}{\epsilon_0 A}\right)$$

Total potential drop:

$$V_a - V_b = E_I \cdot (d - d_1) + E_{II}\, d_1 = \frac{Q_0}{\epsilon_0 A}\left((d-d_1) + \frac{d_1}{k}\right)$$

$$\therefore \boxed{C = \frac{Q_0}{V_a - V_b} = \frac{\epsilon_0 \cdot A}{(d-d_1) + d_1/k}}$$

Method #2: (capacitors in series)



$$C_1 = \frac{\epsilon_0 \cdot A}{d - d_1}$$

$$C_2 = k\,\frac{\epsilon_0 A}{d_1}$$

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{(d-d_1) + d_1/k}{\epsilon_0 A}}$$

• Storage of electrostatic energy in a capacitor with and without dielectric filling

Without dielectric filling.

$$\Delta U = \frac{Q^2}{2C} = \frac{C_0}{2}(V_a - V_b)^2$$

Energy density inside a parallel-plate capacitor is uniform

$$\Delta U = u(Ad) = \frac{1}{2}\left(\frac{\epsilon_0 A}{d}\right)(E \cdot d)^2 = \frac{\epsilon_0}{2}E^2$$

$$\therefore \boxed{u = \frac{\epsilon_0}{2}E^2}$$

True even when E is not uniform.

With dielectric filling,

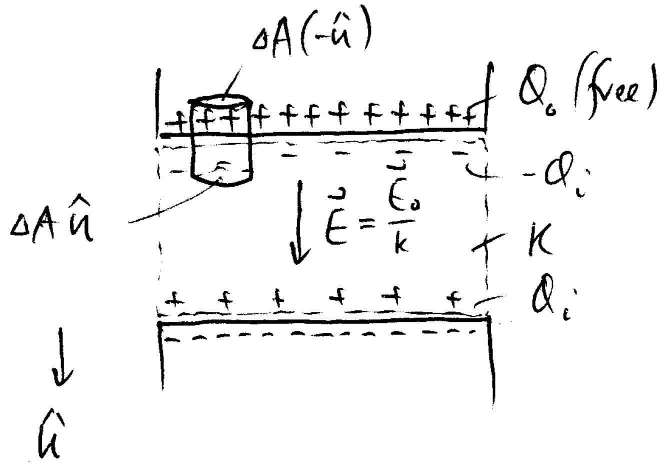
$$\Delta U_K = K\frac{C_0}{2}(V_a - V_b)_K^2 = K \cdot \left(\frac{\epsilon_0}{2}E^2\right) \cdot (Ad)$$

$$\boxed{u_K = \frac{\Delta U_K}{(Ad)} = \frac{K\epsilon_0}{2}E^2}$$

$\boxed{\text{Example 25-10 \& 11}}$

- Gauss' law in dielectrics (optional)

$$\oint\!\!\oint \vec{E}\cdot d\vec{A} = \frac{\Delta A}{\epsilon_0}\left(\sigma_0 - \sigma_i\right)$$



LHS:

$$\oint\!\!\oint \vec{E}\cdot d\vec{A} = \left(\vec{E}_{conductor}\right)\cdot\left(-\Delta A\,\hat{u}\right) + \left(\vec{E}\right)\cdot\left(\Delta A\cdot\hat{u}\right)$$

$$= \left(\cdots\right) + \frac{1}{k}\left(\vec{E}_0\right)\left(\Delta A\cdot\hat{u}\right)$$

By moving $\left(\Delta A/\epsilon_0\right)\left(-\sigma_i\right)$ to LHS:

$$\frac{1}{k}\left(\vec{E}_0\right)\cdot\left(\Delta A\cdot\hat{u}\right) + \left(\frac{\sigma_i}{\epsilon_0}\right)\cdot\Delta A = \frac{1}{k}\left(E_0\,\Delta A\right) + \frac{k-1}{k}E_0\,\Delta A$$

$$= \vec{E}_0\cdot\left(\Delta A\cdot\hat{u}\right) = \left(k\vec{E}\right)\cdot\left(\Delta A\,\hat{u}\right)$$

We have

$$\oint\!\!\oint \left(k\vec{E}\right)\cdot d\vec{A} = \frac{Q_f}{\epsilon_0} \quad \left(\text{since } \vec{E}_{conductor} = 0\right)$$

Generally, we define $\vec{D} = k\epsilon_0\vec{E}$,

$$\oint\!\!\oint \vec{D}\cdot d\vec{A} = Q_f \text{ (inside)}$$

Chapt. 25

Week 13 Electric Current and

Ohm's law for current in homogeneous condensed materials (solids, liquids, etc)

Electric current:

flow of electric charge, typically through a given cross-sectional surface S.

$$I\Big|_{\text{through } S} = \frac{\Delta Q}{\Delta t}\Big|_{\text{through } S \text{ in a given direction}}$$

SI unit: ampere (A) = coulomb/sec

The given direction is specified by the surface normal.

$$I\Big|_{\text{through } S} = \iint_S \vec{J}\cdot(\hat{n}\, dA) = \iint_S \vec{J}\cdot d\vec{A}$$

$\vec{J}$ is the electric current density

$$\vec{J} = |\vec{J}|\,\hat{J} = J\,\hat{J}$$

$\hat{J}$ is the unit vector that specifies the direction of electric current;

$J = |\vec{J}|$ is the absolute magnitude of electric current passing through a unit area that is perpendicular to $\hat{J}$

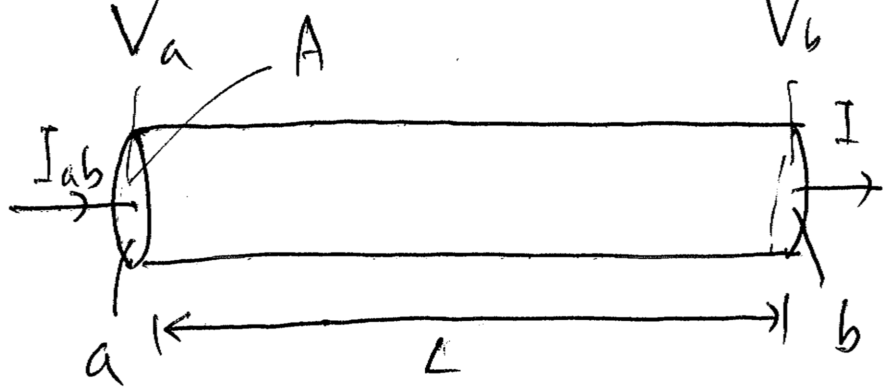
SI unit of J: A/m^2 = ampere$/m^2$

In a solid or a liquid, if electric charged particles have a velocity $\vec{v}_q$, and carry a charge q per particle, and have a volume density n_q then the electric current density contributed by this collection (group) of charged particles (charged carriers) is

$$\vec{J}_q = n_q \cdot q \cdot \vec{v}_q$$

q can be positive or negative; the direction of $\vec{J}$ is determined by the product of q and $\vec{v}_q$. $\vec{v}_q$ can be driven by many forces.

Ohm's law (only electrostatic force)



$$(V_a - V_b) \propto I_{ab} = - I_{ba}$$

Or,
$$(V_a - V_b) = I_{ab} R \quad \text{(Ohm's law)}$$

Proportionality constant:

R: resistance (S.I. unit: ohm $= \frac{\text{Volt}}{\text{Amp}}$)

When L doubles, ΔV needs to be doubled in order to maintain I, thus $R \propto L$;

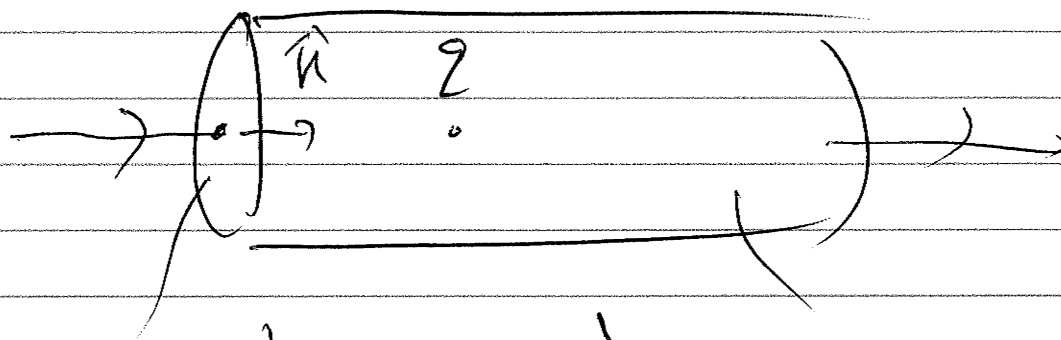
When A doubles, I doubles under same ΔV, thus $R \propto 1/A$:

$$R = \rho \frac{L}{A}$$

ρ: resistivity (ohm·m or $\Omega \cdot$m) material dependent.

Ohm's law: (General forces)

$\vec{\mathcal{E}} \longrightarrow$



$$\vec{F}_{on\,q} = q\vec{\mathcal{E}} \qquad n_q \quad \tau$$

$$m\frac{d\vec{v}_q}{dt} = q\vec{\mathcal{E}} - \frac{m}{\tau}\vec{v}_q$$

$$\Rightarrow \quad q\vec{\mathcal{E}} = \frac{m}{\tau}\vec{v}_q$$

$$\langle\vec{v}_q\rangle = \frac{q\cdot\tau}{m}\vec{\mathcal{E}}$$

$$\vec{J} = n\cdot q\cdot\vec{v}_q = \frac{n_q q^2\cdot\tau}{m}\vec{\mathcal{E}}$$

$$I = A\hat{n}\cdot\vec{J} = \frac{n_q q^2\cdot\tau}{m}\cdot A\,\vec{\mathcal{E}}\cdot\hat{n}$$

$$I_{ab} L = \int_a^b I\, dl = \frac{n_q q^2 \tau A}{m} \int_a^b \vec{\mathcal{E}} \cdot d\vec{l}$$

$$\therefore \int_a^b \vec{\mathcal{E}} \cdot d\vec{l} = I_{ab} \cdot \underbrace{\left(\frac{m}{n_q q^2 \tau}\right) \cdot \frac{L}{A}}_{R}$$

Across a resistor, $\vec{\mathcal{E}} = \vec{E}$

$$V_a - V_b = I_{ab} R$$

$$R = \frac{m}{n_q q^2 \tau} \cdot \frac{L}{A} = \rho \frac{L}{A}$$

Across a battery, $\vec{\mathcal{E}} = \vec{\mathcal{E}}_N + \vec{E}$

$$\int_c^d \left(\vec{\mathcal{E}}_N + \vec{E}\right) \cdot d\vec{l} = I_{cd} r_i$$

$$\int_c^d \vec{\mathcal{E}}_N \cdot d\vec{l} + \left(V_c - V_d\right) = I_{cd} \cdot r_i$$

$$\left| \int_c^d \vec{\mathcal{E}}_N \cdot d\vec{l} \right| \equiv \mathcal{E}$$

If $\vec{\mathcal{E}}_N \cdot d\vec{l} > 0$,

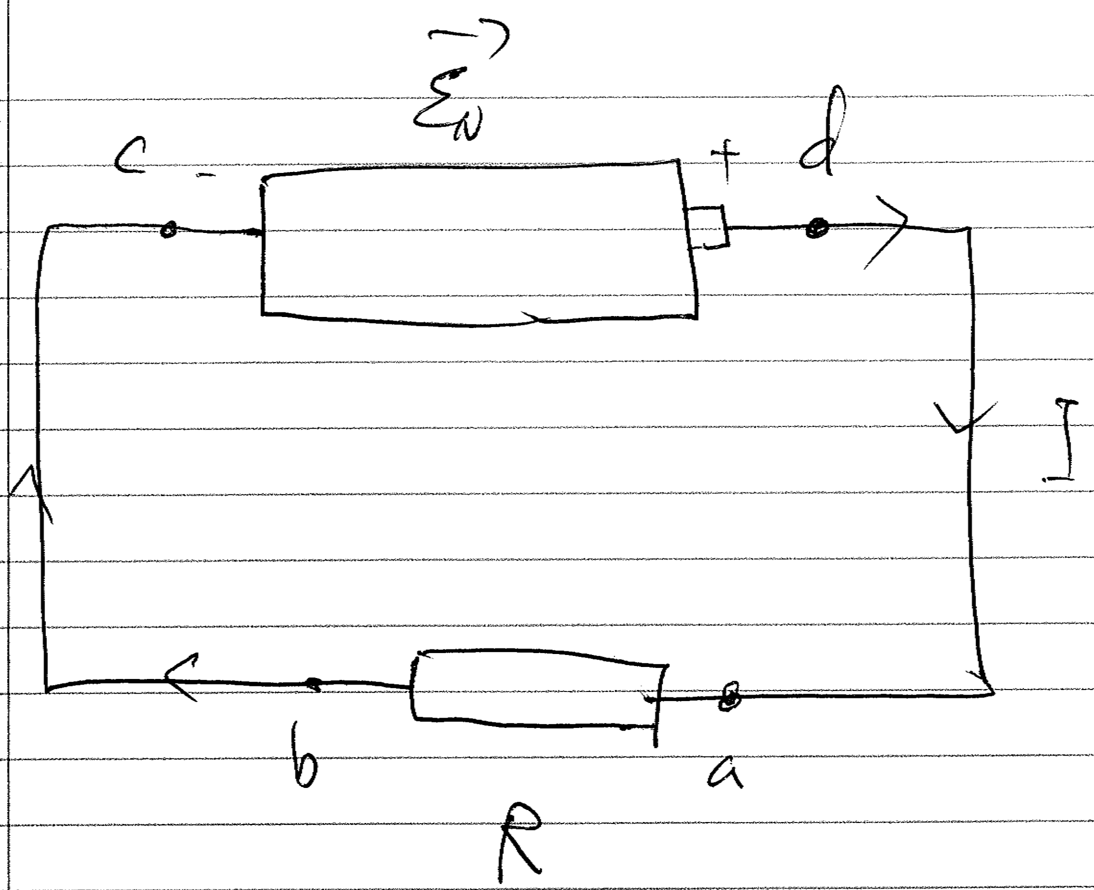
$$\int_{c(-)}^{d(+)} \vec{\mathcal{E}}_N \cdot d\vec{l} = \mathcal{E} = \int_{(-)}^{(+)} \vec{\mathcal{E}}_N \cdot d\vec{l}$$

$$\mathcal{E} + \left(\overset{(-)}{V_c} - \overset{(+)}{V_d}\right) = I_{cd} \cdot r_i$$

Ideal battery, $r_i = 0$; open-circuit battery

$I_{cd} = 0$ or $r_i = 0$

$$\mathcal{E} = \overset{(+)}{V_d} - \overset{(-)}{V_c} > 0 \Rightarrow \overset{(+)}{V_d} > \overset{(-)}{V_c}$$



Steady-current $I_{ab} = I_{cd}$

Across R:

$$I_{ab} R = V_a - V_b$$

Across battery

$$I_{cd} \cdot r_i = \mathcal{E} + (\overset{(-)}{V_c} - \overset{(+)}{V_d})$$

But $\overset{(-)}{V_c} - \overset{(+)}{V_d} = -(V_a - V_b)$

$$\mathcal{E} = I \cdot (r_i + R)$$

$$\Rightarrow I = \frac{\mathcal{E}}{R + r_i}$$

Another look

$$\oint \vec{E}\cdot d\vec{l} = 0$$

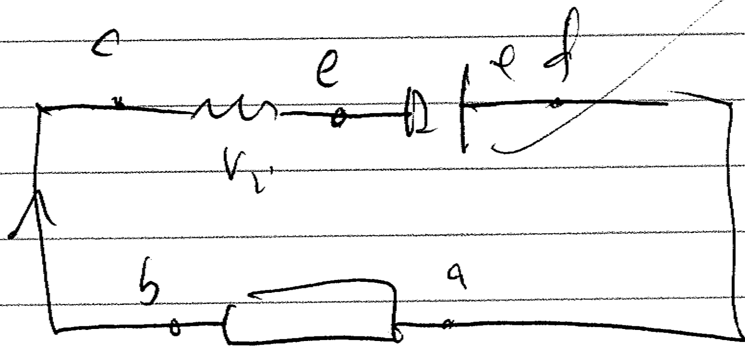
$$\int_a^b \vec{E}\cdot d\vec{l} + \int_{c\,(-)}^{d\,(+)} \vec{E}\cdot d\vec{l} = 0$$

$$\underbrace{\int_a^b \vec{E}\cdot d\vec{l}}_{I_{ab}R} \qquad \underbrace{\int_{c\,(-)}^{d\,(+)} \vec{E}\cdot d\vec{l}}_{I_{cd}r_i - \mathcal{E}}$$

$$\Rightarrow \quad I_{ab}(R + r_i) - \mathcal{E} = 0$$

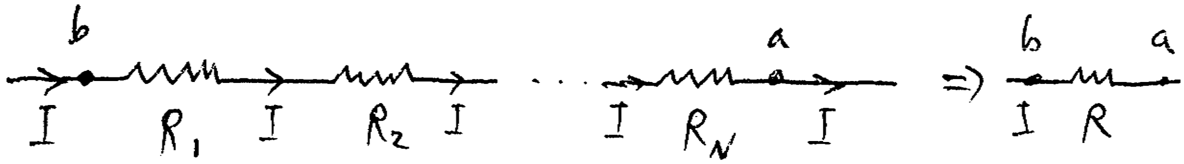
$$I_{ab} = \frac{\mathcal{E}}{R + r_i}$$

$V_d - V_e = \mathcal{E}$



Chapt. 26

- Resisters in series



Equivalent of increasing the total length of a resistive wire, $R = \rho\, L/\Delta S \propto L$, so we expect

$$\boxed{R = R_1 + R_2 + \cdots + R_N}$$

Generally,

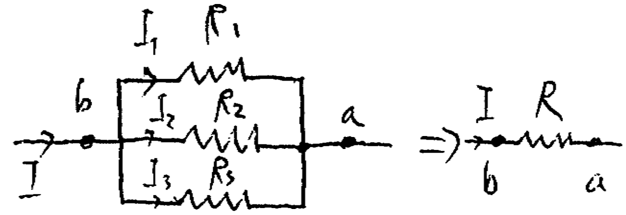
$$V_b - V_a = RI$$

$$R = \frac{V_b - V_a}{I} = \frac{\Delta V_1 + \Delta V_2 + \cdots + \Delta V_N}{I} = R_1 + R_2 + \cdots + R_N$$

- Resisters in parallel.

Equivalent of increasing cross-section area ΔS,

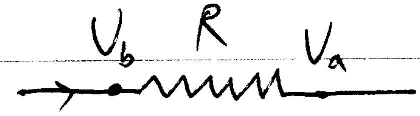
$1/R = \Delta S / \rho L \propto \Delta S$



$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \cdots + \frac{1}{R_N}} \Leftarrow \frac{V_b - V_a}{R} = \frac{V_b - V_a}{R_1} + \frac{V_b - V_a}{R_2} + \cdots$$

- Power dissipation in a resistor R

During dt, electro-static potential energy loss to the resistor (as the charges dQ has gained a net kinetic energy)



$I = dQ/dt$

$$dU = (V_b - V_a)\cdot dQ = (V_b - V_a)\cdot I \cdot dt = V \cdot I \cdot dt$$

The power loss (dissipation) in a resistor R:

$$P = \frac{dU}{dt} = \frac{dW}{dt} = V \cdot I = \frac{V^2}{R} = I^2 R$$

<u>Power generated by an emf source</u>

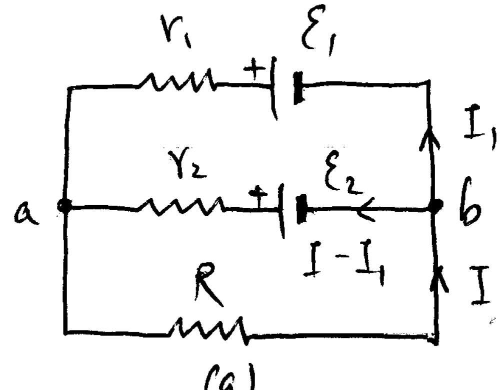
$P_\mathcal{E} = \mathcal{E} \cdot I$ (work done per unit time by a source of emf)

Example 26-9

Example 27-1

• Direct-current Circuits

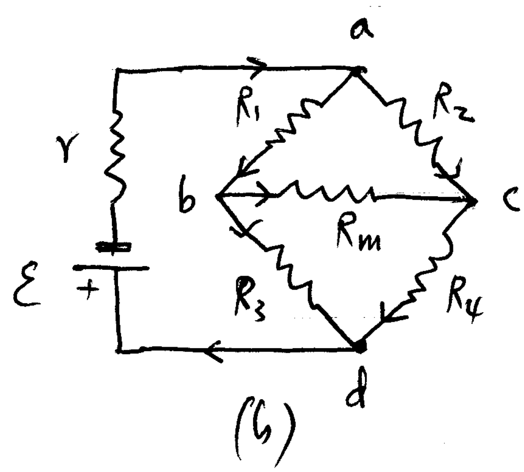
Kirchoff's Rules



Rule #1. (Junction rule)

Algebraic sum of the currents into a junction (where three or more circuit branches meet) is zero so that no net charge is accumulated or drained from the junction

$$\sum I = 0 \quad \text{or} \quad \sum I_{in} = \sum I_{out}$$



Rule #2: (Loop rule)

Algebraic sum of the potential difference (drop or rise) is zero:

$$\oint_c \vec{E} \cdot d\vec{l} = \sum V = 0$$

There are same number of independent loops (including a fresh branch) as of unknown currents, completely solvable.

$$\begin{cases} (R-r_2)I - r_2 I_1 = \mathcal{E}_2 \\ RI - r_1 I = \mathcal{E}_1 \end{cases}$$

Example (a)

Lower loop: $-IR + \mathcal{E}_2 - (I-I_1)\cdot r_2 = 0$

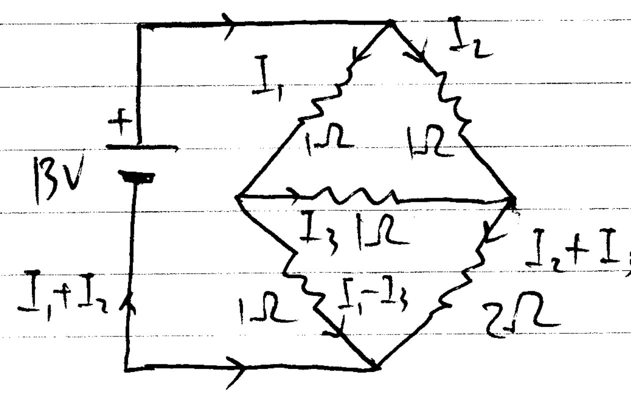
Larger loop: $-IR + \mathcal{E}_1 - I_1 r_1 = 0$

Two unknowns, two equations

Example (b):

Three independent currents to be determined;
Three independent loops that enable setting up three linear equations for solving for the current.

Example 27-6



Kirchoff Rules in practice

1. Assign a current (magnitude & direction) in each branch of the circuit. Apply the Junction Rule so that you only have the minimum number of currents that are independent of each, while currents in other branches can be obtained as a linear combination of these independent currents.

2. Identify an equal number of independent loops that each contains at least one branch of the circuit not shared by other loops, and assign a loop direction.

3. Along the direction of each loop, identify the potential difference, given the orientation of the emf and the assigned current

   Emf $\mathcal{E}$: from low to high $+\mathcal{E}$
   from high to low $-\mathcal{E}$

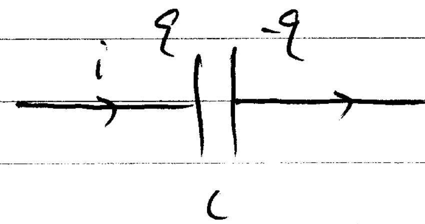
   Resistor R: loop direction along the current : $-IR$
   loop direction against the current $+IR$.

4. When solved $I > 0$, I is along the assigned direction

4. (continue) When solved $I < 0$, I in the branch of interest is along the opposite direction to the assigned one.

5. If a current i (time-varying case) encounters a capacitor C, then put q on the first plate, $-q$ on the second plate, and the potential difference along the assigned current direction is

$$\left(-q/C\right)$$



and

$$i = \frac{dq}{dt}.$$

$q(t=0)$ should be known.

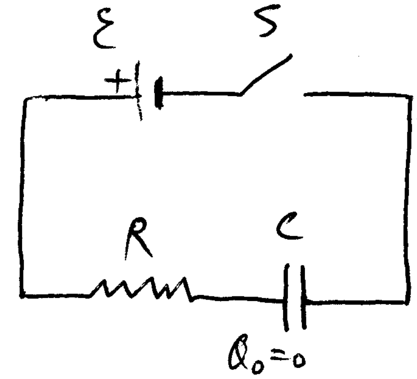
If the loop direction is opposite to the assigned direction of i, then the potential difference is

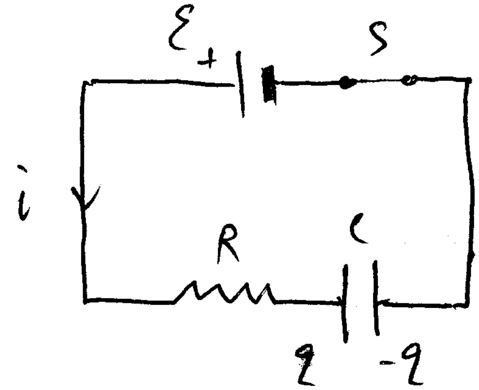
$$\left(q/C\right).$$

- Resistance - capacitance Circuits (RC-Circuits)

Current and potential difference vary with time, but slow enough that Kirchoff Rules apply: charges can only be accumulated on capacitors;

$$\oint_L \vec{E}\cdot d\vec{\ell} = 0$$





When S is closed, along a ccw loop,

$$\begin{cases} -iR - \dfrac{q}{C} + \varepsilon = 0 \\ i = \dfrac{dq}{dt}, \quad q(0) = 0 \end{cases}$$

$$\begin{cases} \dfrac{dq}{dt} + \dfrac{q}{RC} = \dfrac{\varepsilon}{R} \\ q(t=0) = 0 \end{cases} \Rightarrow \frac{d}{dt}(q - \varepsilon C) = -\frac{1}{RC}(q - \varepsilon C)$$

$$q(t) - \varepsilon C = (q(0) - \varepsilon C)e^{-t/RC}$$

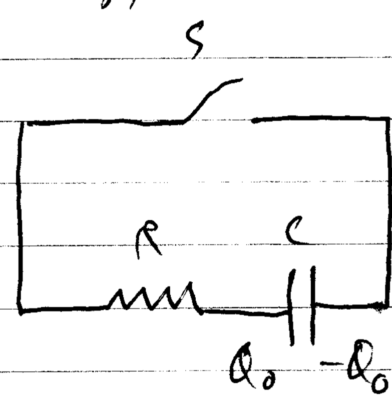
$$\therefore q(t) = \varepsilon C\left(1 - e^{-t/RC}\right), \quad i(t) = \frac{dq}{dt} = \frac{\varepsilon}{R}e^{-t/RC}$$

At $t=0$, $q(0)=0$, the capacitor appears as a short with no resistance, so that the potential on R equals $\mathcal{E}$.

At $t=+\infty$, $q(\infty)=\mathcal{E}C$, or $i(+\infty)=0$, the capacitor appears as an open with infinite resistance, so that the potential drop on R is zero, while the potential drop across C is $\mathcal{E}$.

• Discharging a capacitor & energy conservation

When S is closed,



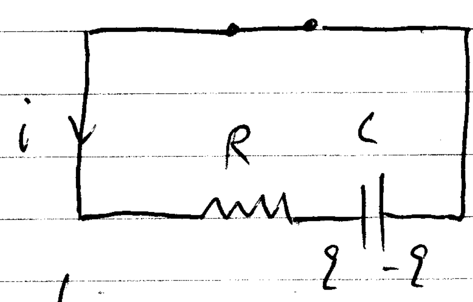
$$\begin{cases} -iR - q/C = 0 \\ i = \dfrac{dq}{dt}, \quad q(0)=Q_0 \end{cases}$$

$$\frac{dq}{dt} = -\frac{q}{RC}$$

$$\Downarrow$$

$$q(t) = Q_0\, e^{-t/RC}$$

$$i = \frac{dq}{dt} = -\frac{Q_0}{RC} e^{-t/RC}$$



flowing in the opposite direction to the assigned.

Time constant $\tau = RC$

Energy stored in the capacitor before discharging

$$U_0 = \frac{Q_0^2}{2C}$$

Energy dissipated in the resistor during discharging

$$\Delta U = \int_0^{+\infty} \frac{dW}{dt} \cdot dt = \int_0^{+\infty} dt \cdot (i^2 R)$$

$$= \int_0^{+\infty} \left(-\frac{Q_0}{RC}\right)^2 \cdot R \cdot e^{-2t/RC} \cdot dt$$

$$= \frac{Q_0^2}{2C} \cdot \int_0^{+\infty} e^{-2t/RC} \, d\left(2t/RC\right)$$

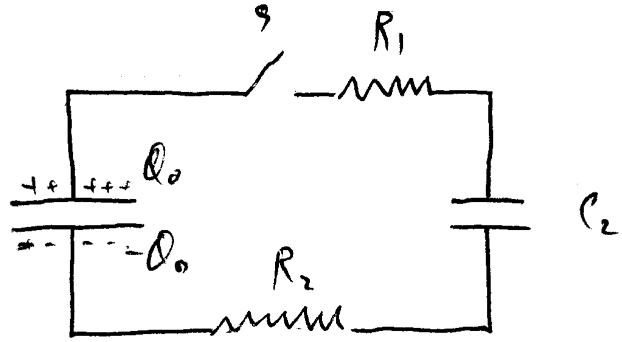
$$= \frac{Q_0^2}{2C}$$

Example: (Optional)

Energy stored $U_0 = \frac{Q_0^2}{2C_1}$

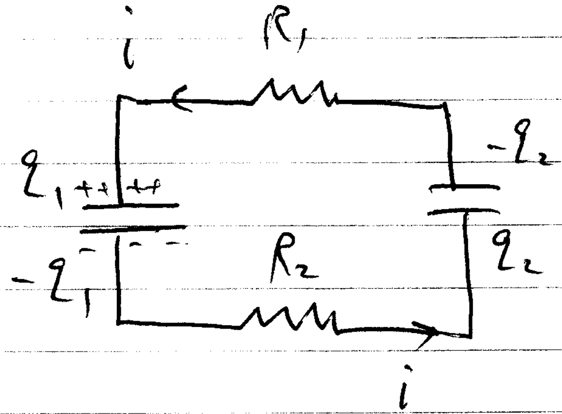
After S is closed,

$$U = \frac{Q_0^2}{2(C_1+C_2)}$$



Energy loss: $U_0 - U = \frac{C_2 \cdot Q_0^2}{2C_1(C_1+C_2)}$

$$\begin{cases} -iR_1 - q_1/C_1 - iR_2 - q_2/C_2 = 0 \\ q_1 - q_2 = Q_0 \\ i = \dfrac{dq_1}{dt} \\ q_1(0) = Q_0 \end{cases}$$



$$\frac{dq_1}{dt}(R_1+R_2) = -\frac{q_1}{C_1} - \frac{q_1 - Q_0}{C_2}$$

$$\frac{dq_1}{dt} = -\frac{(C_1+C_2)}{(R_1+R_2)\cdot C_1 C_2}\cdot\left(q_1 - \frac{C_1}{C_1+C_2}Q_0\right)$$

$$\therefore\ q_1(t) - \frac{C_1}{C_1+C_2}Q_0 = \left(q_1(0) - \frac{C_1}{C_1+C_2}Q_0\right)e^{-\frac{(C_1+C_2)\cdot t}{(R_1+R_2)\cdot C_1\cdot C_2}}$$

$$\therefore\ q_1(t) = \frac{C_1\cdot Q_0}{C_1+C_2} + \frac{C_2 Q_0}{C_1+C_2}e^{-\frac{(C_1+C_2)\cdot t}{(R_1+R_2)\cdot C_1\cdot C_2}}$$

$$i = \frac{dq_1(t)}{dt} = -\frac{Q_0}{(R_1+R_2)\cdot C_1}e^{-(C_1+C_2)\cdot t/(R_1+R_2)\cdot C_1\cdot C_2}$$

$$\Delta U\Big|_{\text{loss in } R_1 \,\&\, R_2} = \int_0^{+\infty} dt\, i^2 (R_1+R_2) = \frac{C_2\cdot Q_0^2}{2C_1(C_1+C_2)}$$ *

Chapter 27. Magnetic forces

- Magnetic forces between permanent "magnets"

Solid materials containing Fe and a number of other elemental materials exert forces on each other, even when carrying no ~~electric~~ net electric charges

They are magnets.

- A magnet of any shape has two opposite ends (or faces) that are called magnetic poles.

When freely suspended, one magnetic pole of a magnet points to the north pole of the Earth, and is called "the north pole" of the magnet. The other magnetic pole points to the south pole of the Earth, and is called "the south pole" of the magnet.

A compass needle is a needle-shape magnet with its two ends chosen to the two magnetic poles.

- Like magnetic poles repel each other

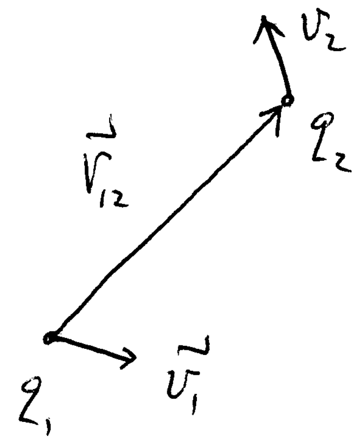
Opposite magnetic poles attract each other

(But how they work is a bit complicated, ...)

- Magnetic force by a moving charge on another moving charge

Lorentz-Biot-Savart law of magnetic force:
(magnetic counterpart of Coulomb's law)

Force of a moving point charge $(q_1, \vec{v}_1)$ on another moving point charge $(q_2, \vec{v}_2)$ at a field position $\vec{r}_{12}$ from the source



$$\vec{F}_{M\,(1 on 2)} = \frac{\mu_0}{4\pi} \frac{q_2 \vec{v}_2 \times (q_1 \vec{v}_1 \times \hat{r}_{12})}{r_{12}^2}$$

Magnetic field $\vec{B}$ produced by a moving point charge $(q_1, \vec{v}_1)$ at a field position $\vec{r}$ from q_1

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{q_1 \vec{v}_1 \times \hat{r}}{r^2} \qquad \text{(S.I. unit: tesla or T)}$$

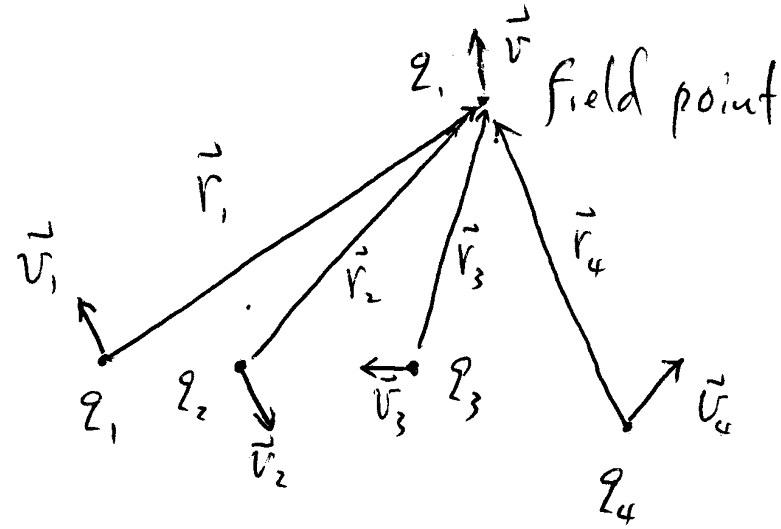
$$\vec{F}_{M\,(1 on 2)} = q_2 \vec{v}_2 \times \vec{B}(\vec{r}) = q_2 \vec{v}_2 \times \vec{B}$$

$$1 \text{ tesla} = 1\,T = 1\,N/A\cdot m = 10^4 \text{ gauss}$$

$$\mu_0 = 4\pi \times 10^{-7}\, N\cdot s^2/c^2 = 4\pi \times 10^{-7}\, N/A^2 = 4\pi \times 10^{-7}\, T\cdot m/A$$

- Magnetic field $\vec{B}$ produced a collection of moving electric charges $(q_1, \vec{v}_1 ; q_2, \vec{v}_2 ; \cdots ; q_N, \vec{v}_N)$

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_1 \vec{v}_1 \times \hat{r}_1}{r_1^2} + \frac{\mu_0}{4\pi}\frac{q_2 \vec{v}_2 \times \hat{r}_2}{r_2^2} + \cdots + \frac{\mu_0}{4\pi}\frac{q_N \vec{v}_N \times \hat{r}_N}{r_N^2}$$

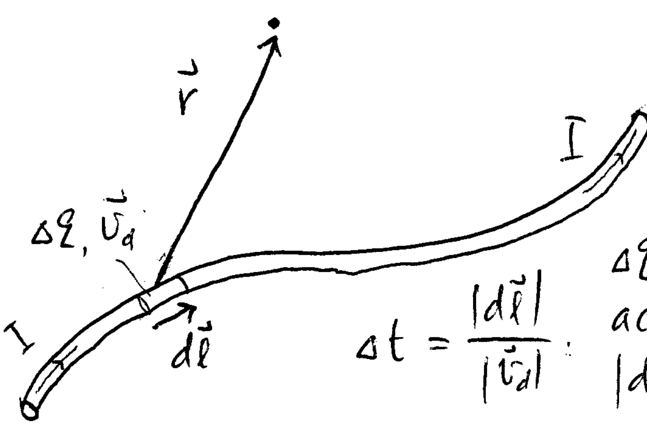


Magnetic force on a moving point charge $(q, \vec{v})$ at the field point

$$\vec{F}_M = q\vec{v} \times \vec{B}$$

- Magnetic field $\vec{B}$ produced by a segment of current-carrying thin wire

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{\Delta q \cdot \vec{v}_d \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{\Delta q \cdot d\vec{l} \times \hat{r}}{\Delta t \cdot r^2} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{l} \times \hat{r}}{r^2}$$



$\Delta t = \frac{|d\vec{l}|}{|\vec{v}_d|}$: Δq sweeps across $|d\vec{l}|$

$$\vec{B} = \int \frac{\mu_0}{4\pi} \cdot \frac{I d\vec{l} \times \hat{r}}{r^2}$$

Magnetic force felt by a moving charge $(q, \vec{v})$ in a magnetic field $\vec{B}$

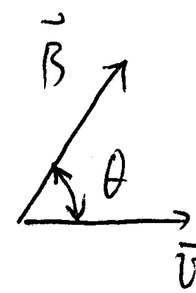
$$\vec{F}_M = q\vec{v} \times \vec{B} = q \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= q(v_y B_z - v_z B_y)\hat{i} + q(v_z B_x - v_x B_z)\hat{j} + q(v_x B_y - v_y B_x)\hat{k}$$

$\vec{F}_M \perp \vec{v}$

$\vec{F}_M \perp \vec{B}$

$|\vec{F}_M| = |q| v B \sin\theta$



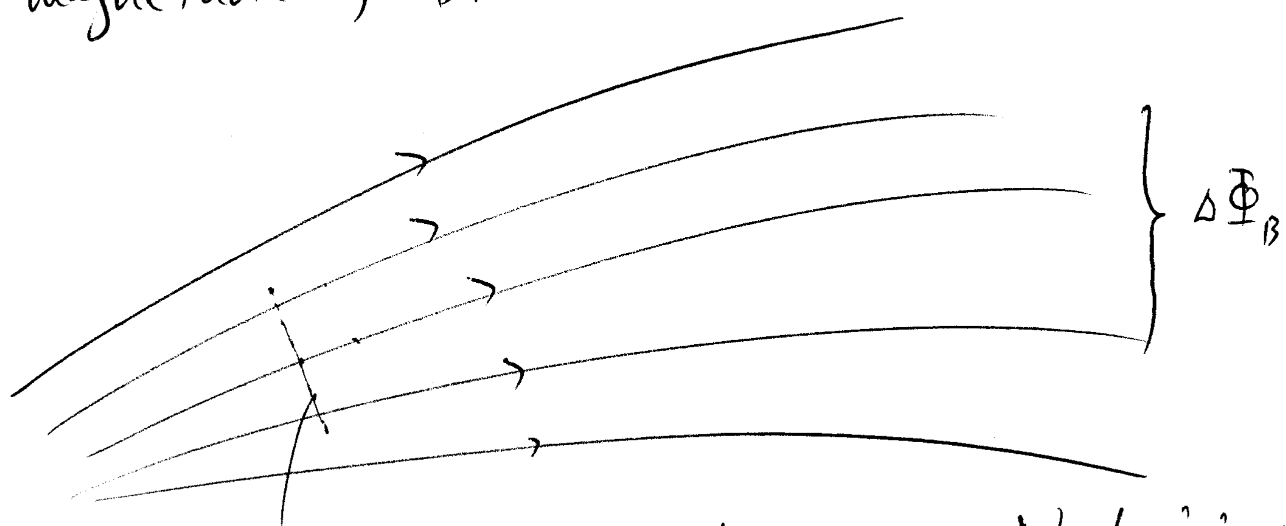
Example 29-1

Example 29-2

Example 28-1

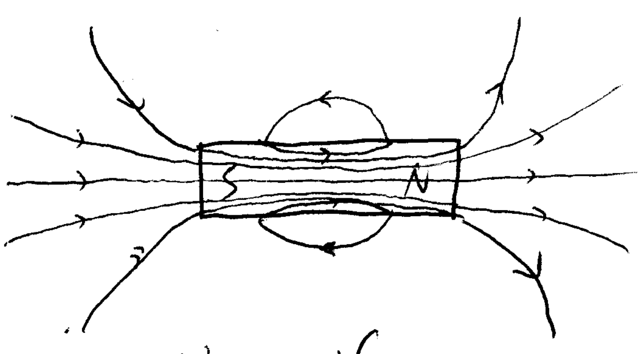
- Magnetic field lines and magnetic flux through a directed (vector) surface element

The distribution of a magnetic field $\vec{B}(\vec{r})$ can be represented by a set of magnetic field lines such that the direction of the lines is along that of $\vec{B}$, and the number density (i.e., number of the field lines crossing a unit surface area that is perpendicular to $\vec{B}$) equals the magnetude of $\vec{B}$.

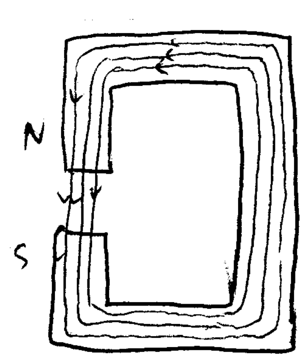


$$\vec{B} = \left|\frac{\Delta \Phi_B}{\Delta A}\right| \cdot \hat{B}$$

No begining!
No end!



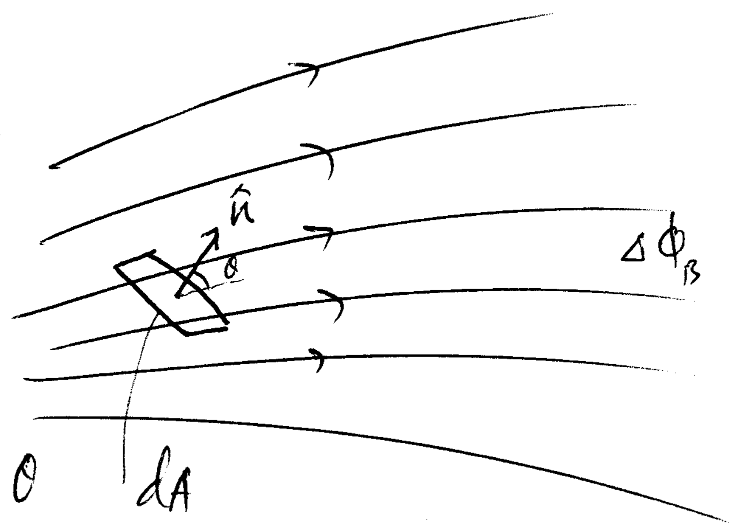
Non-uniform



Uniform

Magnetic flux through a directed (vector) area element $d\vec{A} = dA\,\hat{n}$

— magnetic field lines passing through a directed surface element (cross-section)

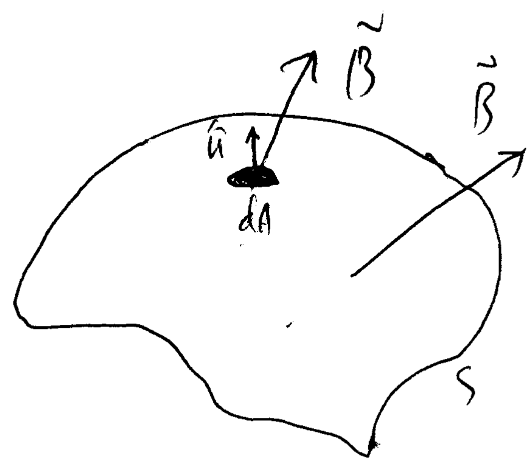


$$d\Phi_B = \vec{B}\cdot d\vec{A} = B\,dA\cdot\cos\theta$$

S.I. unit of Φ_B: webber (Wb) $\quad 1\,\mathrm{Wb} = 1\,\mathrm{T\cdot m^2}$

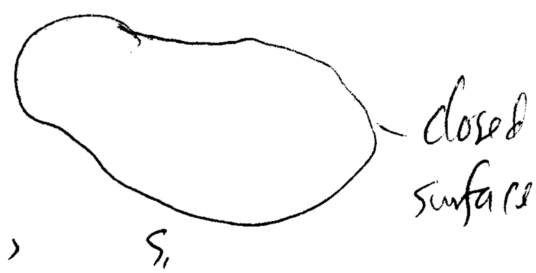
Magnetic flux through an open surface S

$$\Phi_B = \int_S \vec{B}\cdot d\vec{A}$$



<u>Gauss' law of magnetic flux</u>

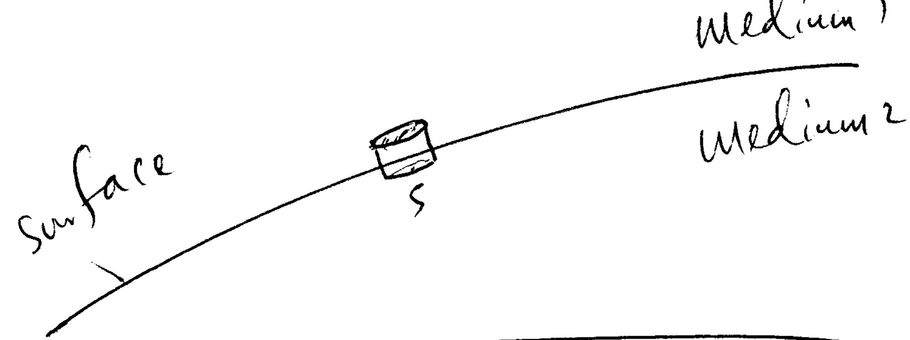
$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$



— no magnetic monopoles

magnetic field lines are closed loops

Unlike Gauss' law of electric flux that we can use under symmetric conditions to determine the magnitude of electric field, we cannot use Gauss' law of magnetic flux to determine $|\vec{B}|$ under symmetric situation.



from $\oint\!\!\oint_s \vec{B}\cdot d\vec{A} = 0$, $\boxed{\vec{B}_1\cdot\hat{n} = \vec{B}_2\cdot\hat{n}}$

- Motion of charged particles in a magnetic field $\vec{B}$ without friction (impedance)

First, since $\vec{F}_M = q\vec{v}\times\vec{B}$, the work done by the magnetic force alone per unit time

$$\frac{dw}{dt} = \vec{F}_M\cdot\vec{v} = q\,\vec{v}\cdot(\vec{v}\times\vec{B}) = 0$$

So $\vec{F}_M$ only changes the direction of a moving charged particle's velocity, not the magnitude of the velocity so that the kinetic energy is conserved in a magnetic field alone.

Charged particle in a constant magnetic field

$$\vec{B} = B_0 \hat{k}$$

Decompose the velocity of a q-charged particle

$$\vec{v} = \vec{v}_\perp + v_\parallel \hat{k} \qquad \vec{v}_\perp \text{ in } x\text{-}y \text{ plane}$$

Since $\vec{F}_M = q\vec{v} \times \vec{B}$ is only in x-y plane

$$\frac{d}{dt}\left(v_\parallel \hat{k}\right) = 0.$$ The particle moves along $\hat{k}$ at a constant speed $v_\parallel$.

In x-y plane, $|\vec{v}_\perp|$ has to be a constant. Acceleration by the magnetic force $\vec{F}_M$ is

$$\left|\vec{F}_M / m\right| = \frac{|q|\, v_\perp B}{m}$$

is also a constant. Thus the particle executes a circular motion, — cyclotron motion, with a radius r such that

$$m \frac{v_\perp^2}{r_c} = |\vec{F}_M| = |q| \cdot v_\perp \cdot B$$

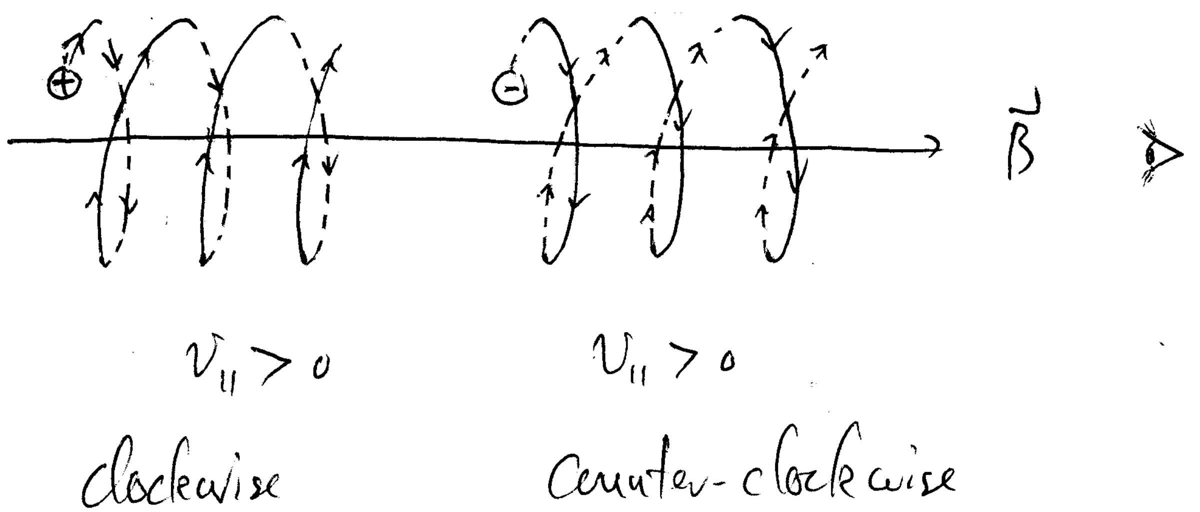
$\therefore$ Cyclotron radius: $r_c = \dfrac{m v_\perp}{|q| \cdot B}$

cyclotron frequency (number of cycles per unit time)

$$f_c = \frac{V_\perp}{2\pi r_c} = \frac{q\cdot B}{2\pi m}$$

Angular cyclotron frequency (radian swept per unit time)

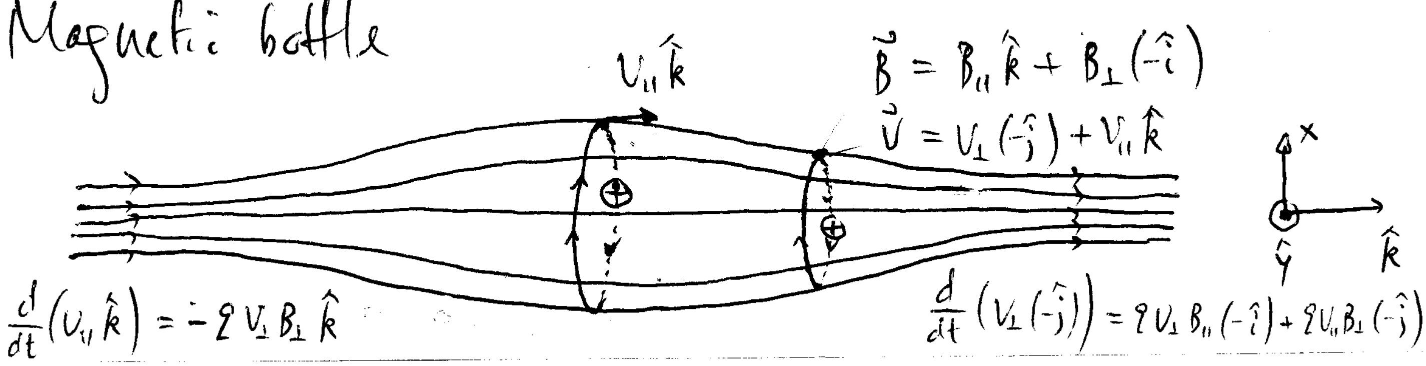
$$\omega_c = 2\pi\cdot f_c = \frac{q\cdot B}{m}$$



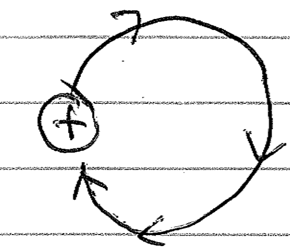
$\boxed{\text{Example 28-4}}$

(Replacing proton with electron, $r_c = R = ?$)

· Magnetic bottle



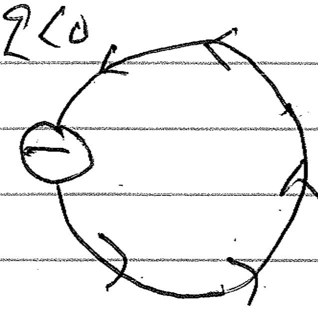
$q>0$



$\vec{B} = B_0 \hat{k}$ $\qquad$ $\vec{B} \parallel \vec{k}$ ⊙

$\vec{v} = \vec{v}_\perp + \vec{v}_z$

$q<0$



$$\frac{d\vec{v}}{dt} = q\,\vec{v} \times \vec{B}$$

$$m\frac{d}{dt}\left(\vec{v}_\perp + \vec{v}_z\right) = q\,\vec{v}_\perp \times \vec{B}$$

$$\Rightarrow \frac{d}{dt}\vec{v}_z = 0, \quad \vec{v}_z = \text{constant}$$

$$\Rightarrow \frac{d}{dt}\vec{v}_\perp = q\,\vec{v}_\perp \times \vec{B}/m$$

$$\Rightarrow \frac{d}{dt}\vec{v}_\perp = \left(-\frac{q}{m}\vec{B}\right) \times \vec{v}_\perp$$

This describes a circular motion with a constant angular velocity

$$\vec{\omega}_c = -\frac{q}{m}\vec{B} \qquad \boxed{\omega_c = \frac{qB_0}{m}}$$

$$\hat{\omega}_c \parallel \vec{k}\,\Big|_{q<0} \; ; \; \vec{\omega}_c \parallel -\hat{k}\,\Big|_{q>0}$$

Since $d\vec{v}_\perp/dt$ is perpendicular to $\vec{v}_\perp$, the magnitude of $\vec{v}_\perp$ is a constant.

Since $\vec{B} \perp \vec{v}_\perp$, $\vec{a} = -(q/m)\vec{B} \times \vec{v}_\perp$ is a constant in magnitude and is always perpendicular to $\vec{v}_\perp$.

Thus $\vec{a}$ is the centripetal acceleration that keeps the particle on a circle of radius R_c and moving a constant linear speed $v_\perp$:

$$\frac{|q|B_0 v_\perp}{m} = \frac{v_\perp^2}{R_c}$$

$$R_c = \frac{m v_\perp}{|q|B_0}$$

$$T_c = \frac{2\pi R_c}{v_\perp} = \frac{2\pi m}{|q| \cdot B_0}$$

$$f_c = \frac{1}{T_c} = \frac{|q| \cdot B_0}{2\pi m}$$

$$\omega_c = \frac{2\pi}{T_c} = 2\pi f_c = \frac{|q| \cdot B_0}{m}$$

Or,

$$\frac{d\vec{U}_\perp}{dt} = \left(-\frac{q}{m}\vec{B}\right) \times \vec{U}_\perp$$

Let $q > 0$, $\vec{\omega}_c = -(q/m)\vec{B} = \left(\frac{qB_0}{m}\right)(-\hat{k}) \equiv \omega_c(-\hat{k})$

$$\begin{cases} \dfrac{dU_x}{dt} = \omega_c U_y \\ \dfrac{dU_y}{dt} = -\omega_c U_x \end{cases}$$

$$\Rightarrow \frac{d^2 U_x}{dt} + \omega_c U_x = 0$$

$$\begin{cases} U_x(t) = U_\perp \cos\omega_c t \\ U_y(t) = \dfrac{1}{\omega_c}\dfrac{dU_x}{dt} = -U_\perp \sin\omega_c t \end{cases}$$

$$\frac{dx}{dt} = U_\perp \cos\omega_c t$$

$$\begin{cases} X(t) = X_0 + \dfrac{U_\perp}{\omega_c}\sin\omega_c t \\ y(t) = y_0 + \dfrac{U_\perp}{\omega_c}\cos\omega_c t \end{cases}$$

Circular motion centered around $\{X_0, y_0)$

with $R_c = U_\perp/\omega_c = \dfrac{mU_\perp}{qB}$ (clockwise)

Let $q<0$, $\tilde{\omega}_c = \frac{|q|\cdot\tilde{B}}{m} = \frac{|q|B_0}{m}\cdot\hat{k} = \omega_c\cdot\hat{k}$

$$\frac{dv_x}{dt} = -\omega_c v_y$$

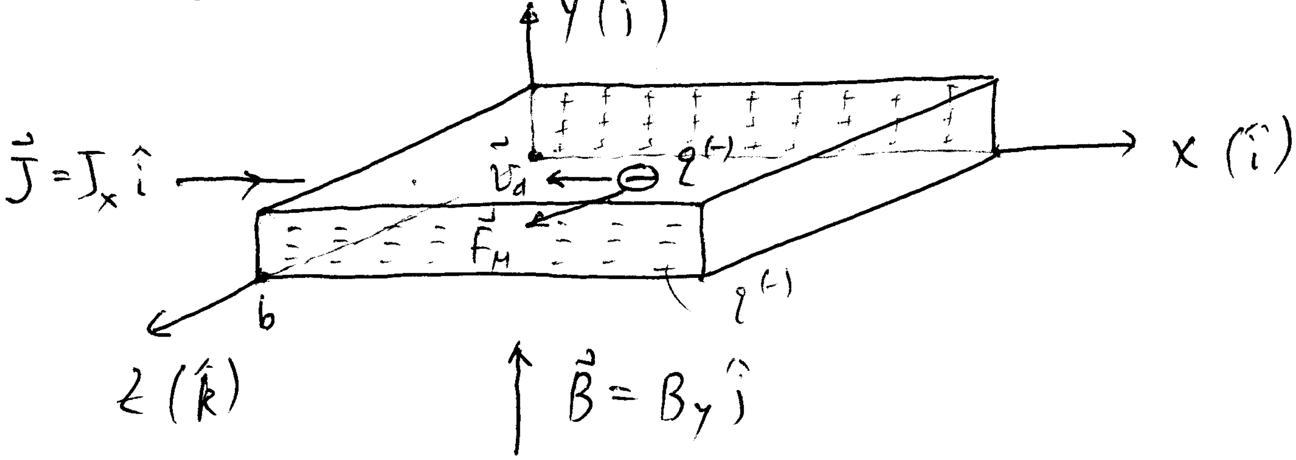
$$\frac{dv_y}{dt} = \omega_c v_x$$

$$\begin{cases} v_x(t) = v_\perp \cos\omega_c t \\ v_y(t) = -\frac{1}{\omega_c}\frac{dv_x}{dt} = v_\perp \sin\omega_c t \end{cases}$$

$$\begin{cases} x(t) = x_0 + \frac{v_\perp}{\omega_c}\sin\omega_c t \\ y(t) = y_0 - \frac{v_\perp}{\omega_c}\cos\omega_c t \end{cases}$$

Again circular motion centered at (x_0, y_0) with radius $R_c = v_\perp/\omega_c = mv_\perp/|q|\cdot B_0$, counter-clockwise, with a constant angular velocity $\omega_c = |q|B/m$ and thus a constant linear velocity $v = \omega_c R_c = v_\perp$.

• Impeded motion of charged particles in a constant magnetic field: Hall Effect



Magnetic force on the negative charge carrier $q^{(-)}$

$$\vec{F}_M = q^{(-)}\,\vec{v}_d^{(-)} \times \vec{B} = |q^{(-)}\,\vec{v}_d|\,B_y \cdot \hat{k}$$

causing $q^{(-)}$ to pile up at the surface $z = b$, and leaving equal amount of positive charge on the opposite surface, $z = 0$. This flow of $q^{(-)}$ stops when the electrostatic force produced by the accumulated charges cancel the magnetic force.

$$\vec{F}_{(\text{on } q^{(-)})} = q^{(-)}\left(\vec{E}_H + \vec{v}_d^{(-)} \times \vec{B}\right) = 0$$

Hall electric field

$$\vec{E}_H = -\vec{v}_d^{(-)} \times \vec{B} = +|\vec{v}_d^{(-)}| \cdot B_y \cdot \hat{k}$$

Hall voltage

$$V_H = V(z=b) - V(z=0) = -|v_d^{(-)}| \cdot B_y \cdot b < 0$$

Hall coefficient

$$R_H \equiv \frac{E_z}{J_x \cdot B_y} = \frac{+|\vec{v}_d^{(-)}| \cdot B_y}{|q^{(-)}| n^{(-)} \cdot |\vec{v}_d^{(-)}| \cdot B_y} = \frac{(-1)}{q^{(-)} n^{(-)}} > 0$$

If $q^{(+)}$ is the charge carrier, $n^{(+)}$ is the density

$$\vec{F}_M (\text{on } q^{(+)}) = q^{(+)} \vec{v}_d^{(+)} \times \vec{B} = |q^{(+)} \vec{v}_d^{(+)}| \cdot B_y \hat{k},$$

thus piling up $q^{(+)}$ at $z = b$, while leaving negative charge at $z = 0$.

Hall Electric field. $\vec{E}_H (q^{(+)}) = -|\vec{v}_d^{(+)}| \cdot B_y \cdot \hat{k} = E_z \hat{k}$

Hall voltage: $V_H = V(z=b) - V(z=0) = |\vec{v}_d^{(+)}| \cdot B_y \cdot b > 0$

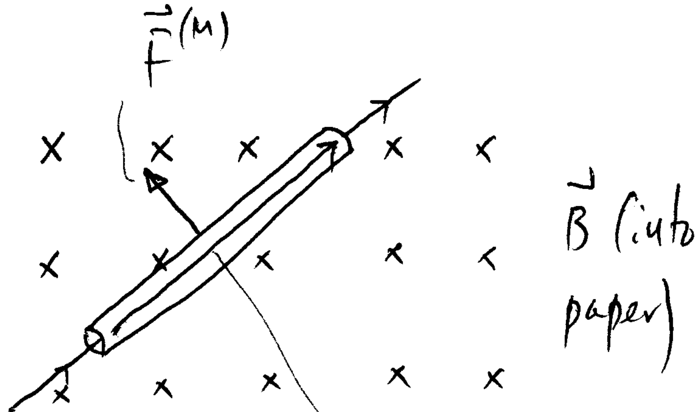
Hall coefficient: $R_H = \dfrac{(-1)}{q^{(+)} \cdot n^{(+)}} < 0$

V_H & R_H reveal charge type and charge density n!

- Magnetic force on current-carrying conductors

Magnetic force on a straight current-carrying segment $(I, \vec{l})$ in a uniform magnetic field $\vec{B}$

$$\boxed{\vec{F}^{(M)} = \Delta q\, \vec{v}_d \times \vec{B}}$$



If electrons carry the current, $\Delta q^{(-)} < 0$,

$$\vec{v}_d^{(-)} = v_d^{(-)}(-\hat{l})$$

$$I = -\frac{\Delta q^{(-)}}{\Delta t}$$

$$I = \frac{\Delta Q}{\Delta t}$$

$$\Delta t = \frac{l}{v_d}$$

$$\vec{F}^{(M)} = \Delta q^{(-)} \vec{v}_d^{(-)} \times \vec{B}$$

$$= \left(-\frac{\Delta q^{(-)}}{\Delta t}\right)\left(v_d^{(-)}(-\hat{l}) \cdot \frac{l}{v_d^{(-)}} \cdot (-)\right) \times \vec{B}$$

$$= I\vec{l} \times \vec{B} \quad \#$$

If positive charges carry the current, $\Delta q^{(+)} > 0$, $\vec{v}_d^{(+)} = v_d^{(+)}\hat{l}$

$I = \frac{\Delta q^{(+)}}{\Delta t}$, thus $\vec{F}^{(M)} = \Delta q^{(+)} \vec{v}_d^{(+)} \times \vec{B} = \left(\frac{\Delta q^{(+)}}{\Delta t}\right)\left(v_d^{(+)}\hat{l} \cdot \frac{l}{v_d^{(+)}}\right) \times \vec{B}$

$$= I\vec{l} \times \vec{B} \quad \#$$

Generally, regardless what carries the current I,

$$\boxed{\vec{F}^{(M)}_{\text{on }\vec{\ell}} = I\vec{\ell}\times\vec{B}}$$

Magnetic force on a curved current-carrying wire in a non-uniform magnetic field $\vec{B}$

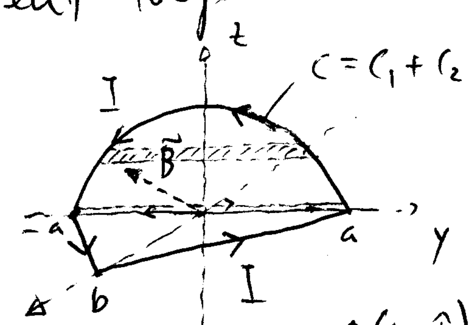
$$d\vec{F}^{(M)}_{\text{on }d\vec{\ell}} = I\,d\vec{\ell}\times\vec{B}$$ (locally straight $d\vec{\ell}$ in a locally uniform $\vec{B}$)

$$\boxed{\vec{F}^{(M)} = \int d\vec{F}^{(M)} = \int I\,d\vec{\ell}\times\vec{B}}$$

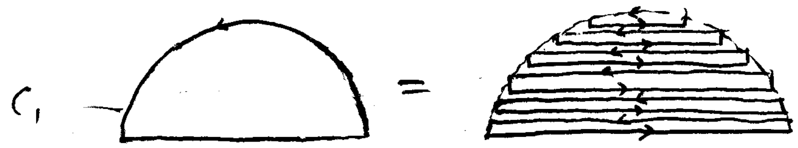
Example 28-8

- Magnetic force and torque on a current loop

$$\vec{F}^{(M)} = \oint_C I\,d\vec{\ell}\times\vec{B} = \oint_{C_1} + \oint_{C_2}$$

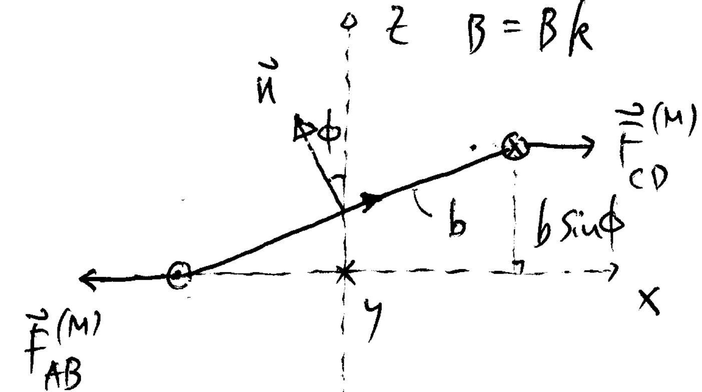


$$\vec{B} = \frac{B(\hat{i}+\hat{k})}{\sqrt{2}}$$



Rectangular current-loop $(a \times b)$ in a uniform $\vec{B}$

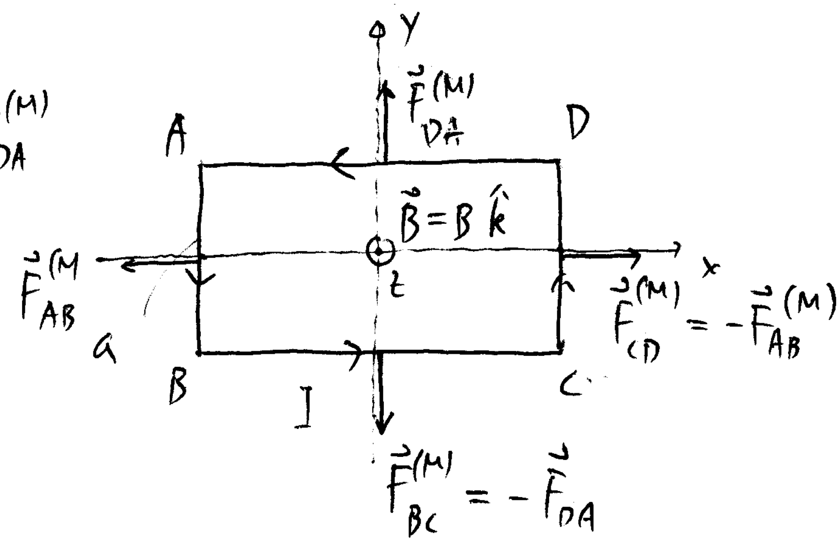
$\hat{n}$: normal unit vector for the rectangular loop chosen by the right-hand rule



$$\vec{F}^{(M)} = \oint I\, d\vec{\ell} \times \vec{B}$$

$$= \vec{F}^{(M)}_{AB} + \vec{F}^{(M)}_{BC} + \vec{F}^{(M)}_{CD} + \vec{F}^{(M)}_{DA}$$

$$= 0$$



Net force on a current loop in a uniform $\vec{B}$ is zero.

(Top-view)

But $\vec{F}^{(M)}_{AB}$ and $\vec{F}^{(M)}_{CD}$ are along two different lines that are apart by $b\sin\phi$, thus exert a torque on the loop

$$|\vec{\tau}| = |\vec{F}^{(M)}_{CD}| \cdot b \cdot \sin\phi = I \cdot a \cdot B \cdot b \cdot \sin\phi = (Iab) \cdot B \cdot \sin\phi$$

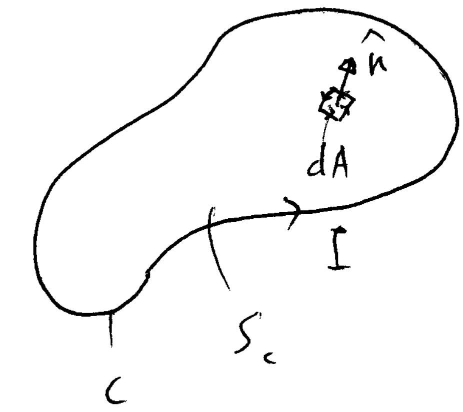
along the direction of $\hat{n} \times \vec{B}$. Define magnetic moment $\vec{\mu} = I(ab\hat{n})$,

$\vec{\tau} = \vec{\mu} \times \vec{B}$ (like $\vec{\tau} = \vec{p} \times \vec{E}$ for an electric dipole)

An arbitrary current-loop (c) in a uniform $\vec{B}$

$$\vec{F}^{(M)} = \oint_c I\, d\vec{l} \times \vec{B} = 0$$

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

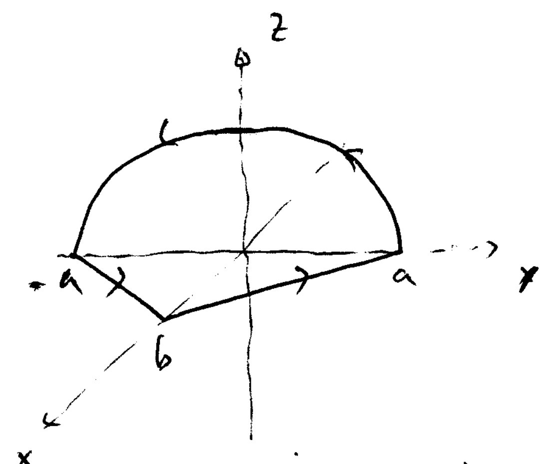


the magnetic dipole moment $\vec{\mu}$ is defined and computed as follows:

Let S_c be a surface that covers the current loop, and an area element $d\vec{A} = \hat{n}\, dA$ is chosen by the right-hand rule

$$\vec{\mu} = \iint_{S_c} I\, d\vec{A} = I \iint_{S_c} d\vec{A} = I \vec{A}$$

Example:



$$\vec{\mu} = I \iint_{S_c} d\vec{A} = I\left(\vec{A}_1 + \vec{A}_2\right)$$

$$= I\left(\frac{\pi}{2} a^2 \hat{i} + ab\, \hat{k}\right)$$

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \frac{IB}{\sqrt{2}}\left(ab - \frac{\pi}{2} a^2\right)\hat{j}$$

$$\vec{B} = \frac{B}{\sqrt{2}}\left(\hat{i} + \hat{k}\right)$$

*

Proof of two equivalent definition of magnetic dipole moment $\vec{\mu}$

$$\vec{\mu} = I \iint_{S_c} d\vec{A} = I \iint_{S_c} \hat{n}\, dA \quad \cdots \text{(1)}$$

$$\vec{\mu} = \frac{I}{2} \oint_c \vec{r} \times d\vec{l} \quad \cdots \text{(2)}$$

Let $\hat{t}$ be a unit vector, we only need to prove

$$\hat{t} \cdot \iint_{S_c} \hat{n}\, dA = \hat{t} \cdot \left(\frac{1}{2} \oint_c \vec{r} \times d\vec{l} \right) \quad \cdots \text{(3)}$$

From the right-hand side

$$\hat{t} \cdot \left(\frac{1}{2} \oint_c \vec{r} \times d\vec{l} \right) = \oint_c \left(\frac{1}{2} \hat{t} \times \vec{r} \right) \cdot d\vec{l}$$

Using the Stokes Theorem.

$$\oint_c \vec{f} \cdot d\vec{l} = \iint_{S_c} (\nabla \times \vec{f}) \cdot d\vec{A}$$

and

$$\nabla \times (\hat{t} \times \vec{r}) = (\vec{r} \cdot \nabla)\hat{t} - (\nabla \cdot \hat{t})\vec{r} - (\hat{t} \cdot \nabla)\vec{r} + (\nabla \cdot \vec{r}) \cdot \hat{t}$$

$$= 2\hat{t}$$

thus

$$\oint_c \left(\frac{1}{2}\hat{t} \times \vec{r}\right) \cdot d\vec{l} = \frac{1}{2} \iint_{S_c} \nabla \times (\hat{t} \times \vec{r}) \cdot d\vec{A}$$

$$= \iint_{S_c} \hat{t} \cdot d\vec{A}$$

$$= \hat{t} \cdot \iint_{S_c} d\vec{A}$$

Thus (3) is valid. Thus (1) = (2)

General proof of $\vec{\tau} = \vec{\mu} \times \vec{B}$ for an arbitrary current loop in a constant magnetic field $\vec{B}$

$$\vec{\mu} = I \iint_{S_c} d\vec{A} = \iint_{S_c} \hat{n}\, dA \cdot I$$

By definition,

$$\vec{\tau} = \oint_c \vec{r} \times (I d\vec{\ell} \times \vec{B})$$

$$= \oint_c I d\vec{\ell}\, (\vec{r} \cdot \vec{B}) - \oint_c I\vec{B}\, (\vec{r} \cdot d\vec{\ell})$$

But $\vec{r} \cdot d\vec{\ell} = \frac{1}{2} d(r^2)$, thus

$$\oint_c I\vec{B}(\vec{r} \cdot d\vec{\ell}) = I\vec{B} \oint_c d(r^2/2) = 0$$

$$\therefore\ \vec{\tau} = \oint_c I d\vec{\ell}\, (\vec{r} \cdot \vec{B})$$

Using $\oint_c \psi\, d\vec{\ell} = \iint_{S_c} d\vec{A} \times (\nabla \psi)$ and $\psi \equiv \vec{r} \cdot \vec{B}$

and $\nabla(\vec{a} \cdot \vec{b}) = (\vec{a} \cdot \nabla)\vec{b} + (\vec{b} \cdot \nabla)\vec{a} + \vec{a} \times (\nabla \times \vec{b}) + \vec{b} \times (\nabla \times \vec{a})$

Now

$$\nabla(\vec{r}\cdot\vec{B}) = \cancelto{0}{(\vec{r}\cdot\nabla)\vec{B}} + (\vec{B}\cdot\nabla)\vec{r} + \cancelto{0}{\vec{r}\times(\nabla\times\vec{B})}$$

$$+ \cancelto{0}{\vec{B}\times(\nabla\times\vec{r})}$$

$$= \vec{B}$$

thus

$$\vec{\tau} = \oint_c I\,d\vec{l}(\vec{r}\cdot\vec{B})$$

$$= I\iint d\vec{A}\times(\nabla(\vec{r}\cdot\vec{B}))$$

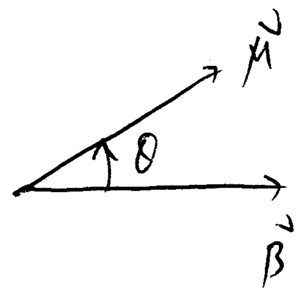
$$= \iint I\,d\vec{A}\times\vec{B}$$

$$= \vec{\mu}\times\vec{B}$$

✱

Potential energy of a magnetic moment $\vec{\mu}$ in a uniform magnetic field

$$\boxed{U^{(m)} = -\vec{\mu}\cdot\vec{B}} \quad \left(\text{like } U^{(E)} = -\vec{p}\cdot\vec{E}\right)$$



$$U^{(m)}(\theta) - U^{(m)}(0)$$

$$= \int_0^\theta \mu B \sin\theta \, d\theta$$

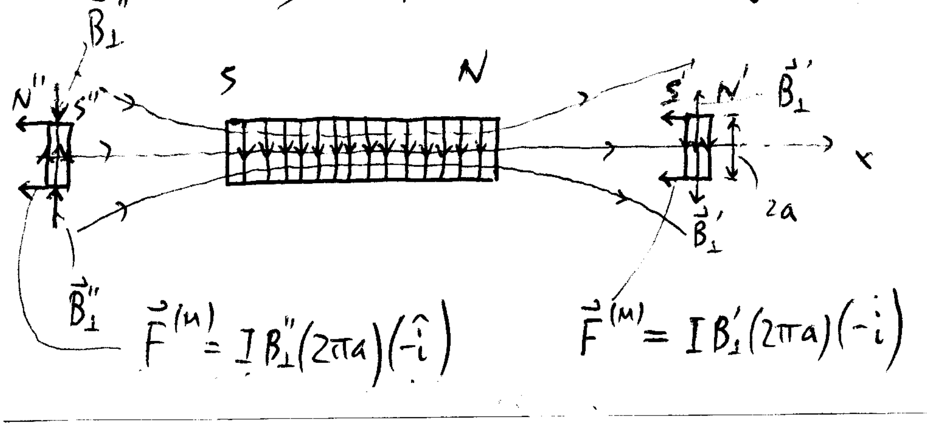
$$= \mu B (1 - \cos\theta)$$

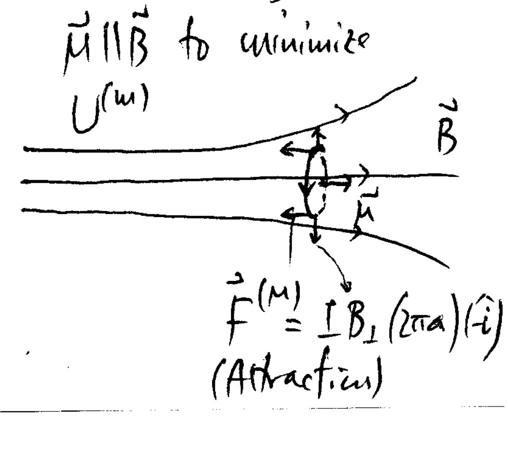
Example 28-9

Example 28-10

- Magnetic dipoles in non-uniform magnetic fields $\vec{B}$

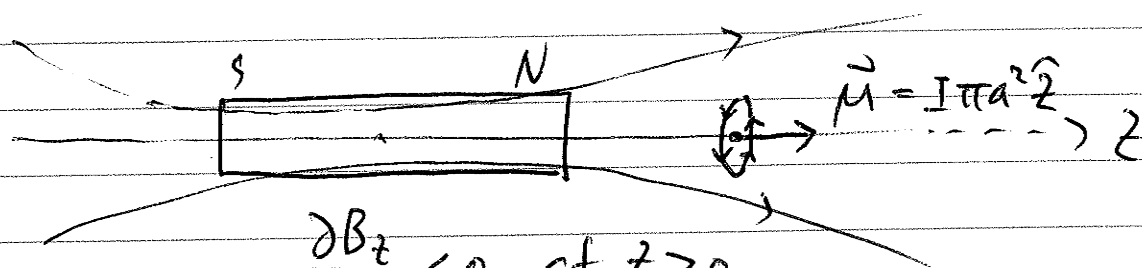
Working of permanent magnets and "induced" magnets





Equivalence of magnetic force of a non-uniform magnetic field $\vec{B}$ on a magnetic dipole moment $\vec{\mu} = I\vec{A}$

Method #1: Potential energy approach



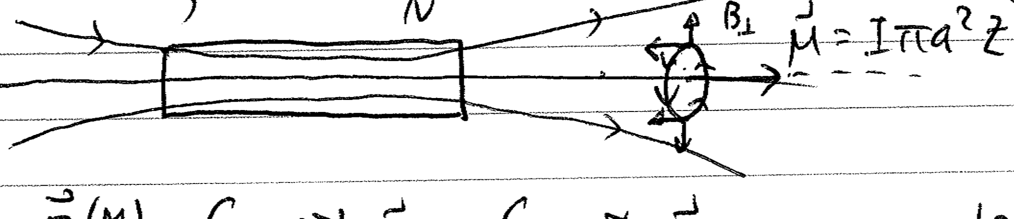
$$\frac{\partial B_z}{\partial z} < 0 \quad \text{at } z > 0$$

$$U^{(m)} = -\vec{\mu}\cdot\vec{B} = -\pi a^2 I\, B(z)$$

$$\vec{F}^{(M)} = -\nabla U^{(m)} = +\hat{z}\frac{\partial}{\partial z}\left(\pi a^2 I\, B(z)\right)$$

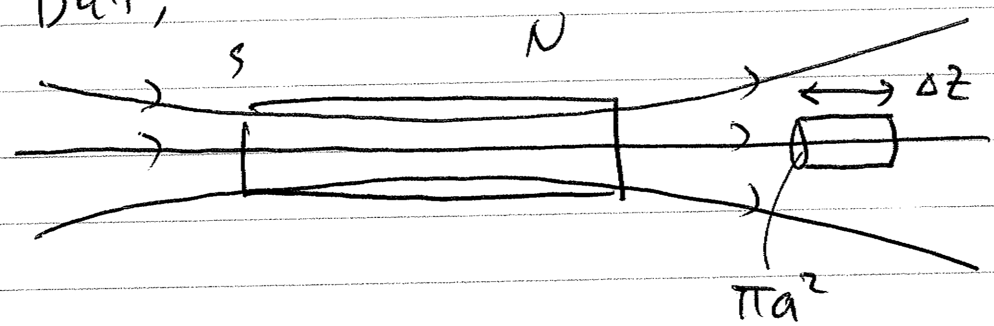
$$= I\pi a^2 \left(\frac{\partial B}{\partial z}\right)\hat{z} = -I\pi a^2 \left|\frac{\partial B}{\partial z}\right| \cdot \hat{z}$$

Method #2: Lorentz force by $\vec{B}_\perp$



$$\vec{F}^{(M)} = \oint I\, d\vec{\ell} \times \vec{B} = \oint I\, d\vec{\ell} \times \vec{B}_\perp = -2\pi a I |B_\perp|\, \hat{z}$$

But,



$$\oint\!\!\!\oint_{\text{cylinder}} \vec{B}\cdot d\vec{A} = 0 = \pi a^2\left(B_z(z+\Delta z) - B_z(z)\right) + 2\pi a\,\Delta z\, B_\perp(z)$$

Thus

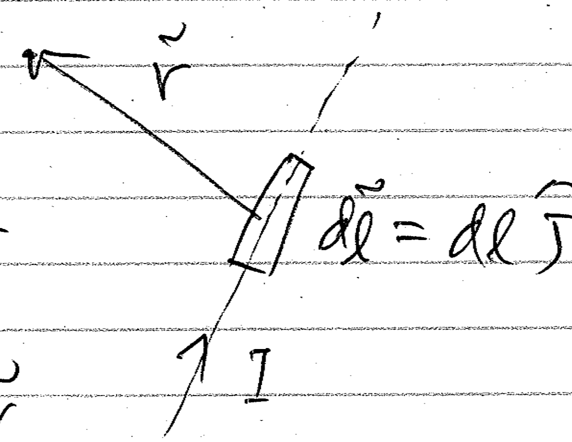
$$-\frac{\partial B}{\partial z} = \frac{2}{a} B_\perp(z)$$

or

$$\left|\frac{\partial B}{\partial z}\right| = \frac{2}{a}\left|B_\perp(z)\right|.$$

As a result, the two methods yield the same result, but from two "refreshing" angles.

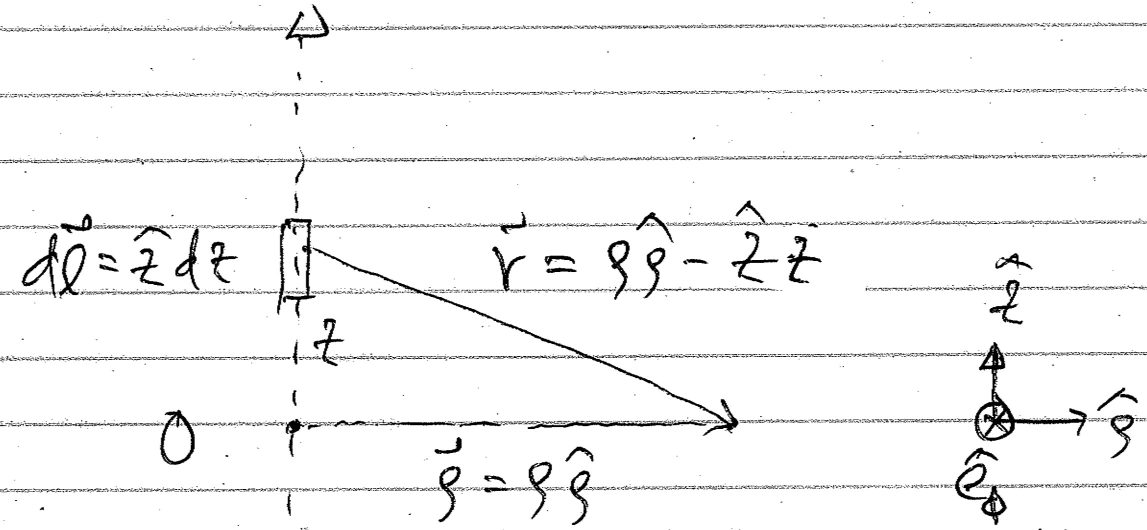
Chapt. 28. Calculation of $\vec{B}$ field from current-carrying wire



$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0 I}{4\pi}\cdot\frac{d\vec{l}\times\vec{r}}{r^3}$$

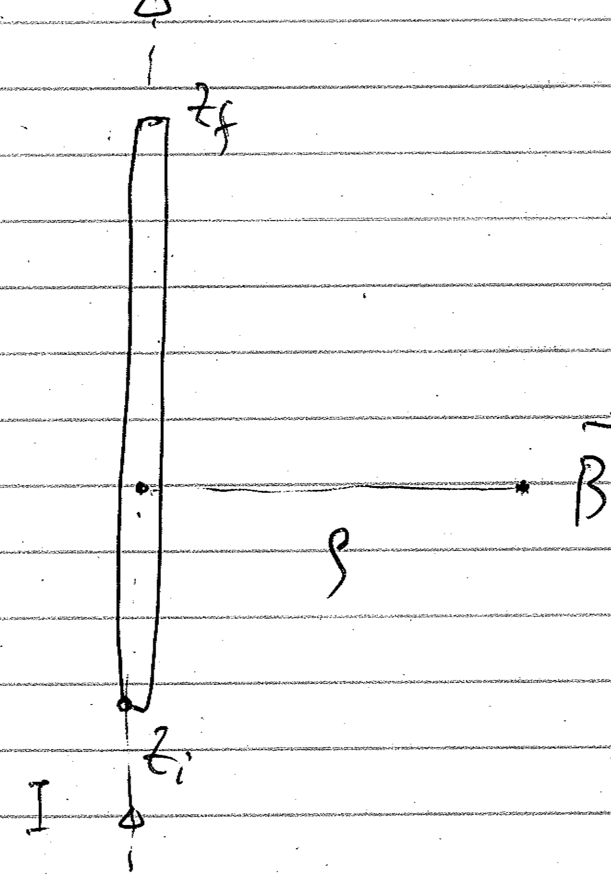
Choose $\hat{J} \parallel d\vec{l}$ along positive z-axis.



$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\cdot\frac{(dz\,\hat{z})\times(\rho\hat{\rho} - z\hat{z})}{r^3}$$

$$= \frac{\mu_0 I}{4\pi}\cdot\frac{\rho\cdot dz}{(z^2+\rho^2)^{3/2}}\cdot\hat{e}_\phi$$

$\vec{B}(\rho)$ from a finite straight segment of current-carrying wire



$$\vec{B}(\rho) = \int_{z_i}^{z_f} d\vec{B} = \frac{\mu_0 I \rho}{4\pi}\,\hat{e}_\phi \int_{z_i}^{z_f} \frac{dz}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi \cdot \rho}\,\hat{e}_\phi \cdot \left(\frac{z_f}{\sqrt{z_f^2+\rho^2}} - \frac{z_i}{\sqrt{z_i^2+\rho^2}} \right)$$

Special cases: Infinitely long straight wire ($z_f = +\infty$, $z_i = -\infty$)

$$\vec{B}(\rho)\Big|_{\text{infinite}} = \frac{\mu_0 I}{2\pi\rho}\hat{e}_\phi$$

Ampere's law

$$\oint_C \vec{B}\cdot d\vec{\ell} = \oint_C \vec{B}\cdot\left(\hat{z}\,dz + \hat{\rho}\,d\rho + \rho\, d\phi\, \hat{e}_\phi\right)$$

$$= \frac{\mu_0 I}{2\pi}\oint_C \left(\frac{1}{\rho}\right)\cdot\left(\rho\, d\phi\right)$$

$$= \mu_0 I$$

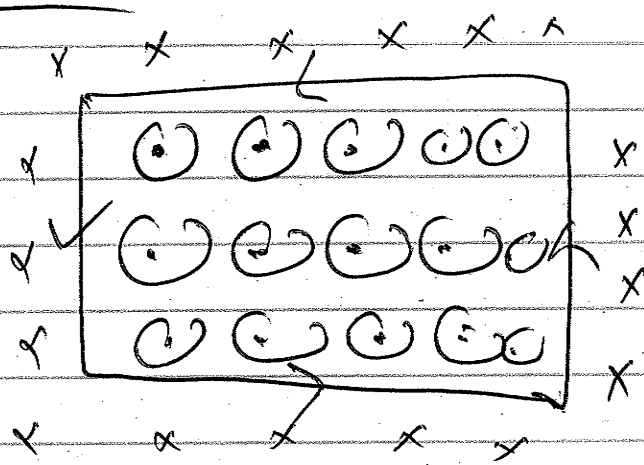
$$= \iint_{S_C} \mu_0 \vec{J}\cdot d\vec{A}$$

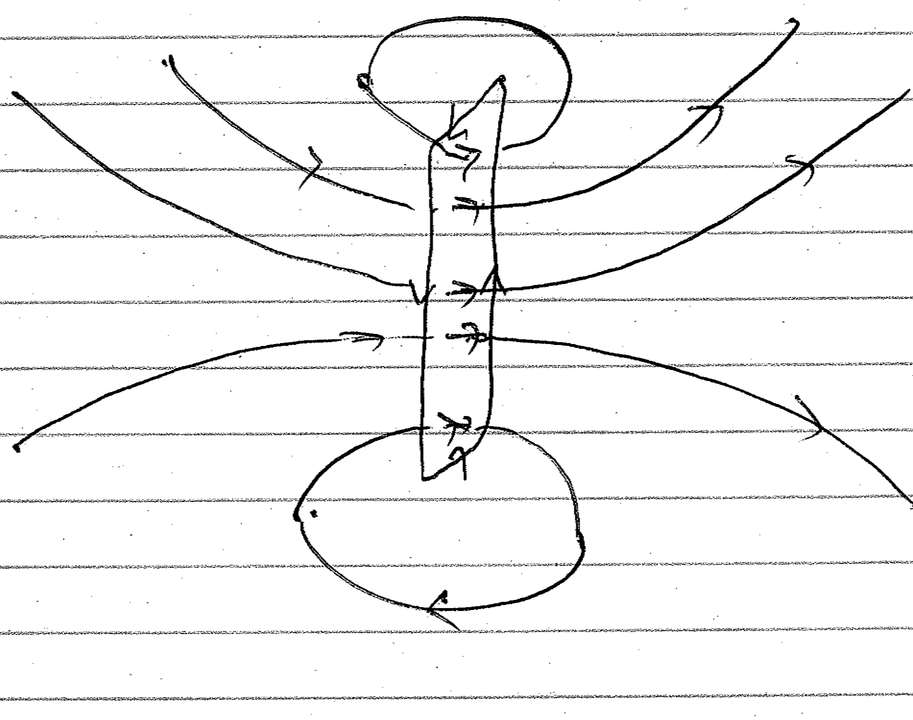
$$= \mu_0 \iint_{S_C} \vec{J}\cdot d\vec{A}$$

$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I\Big|_{\text{through } C} = \mu_0 \iint_{S_C} \vec{J}\cdot d\vec{A}\Big|_{\text{current through } S_C}$$

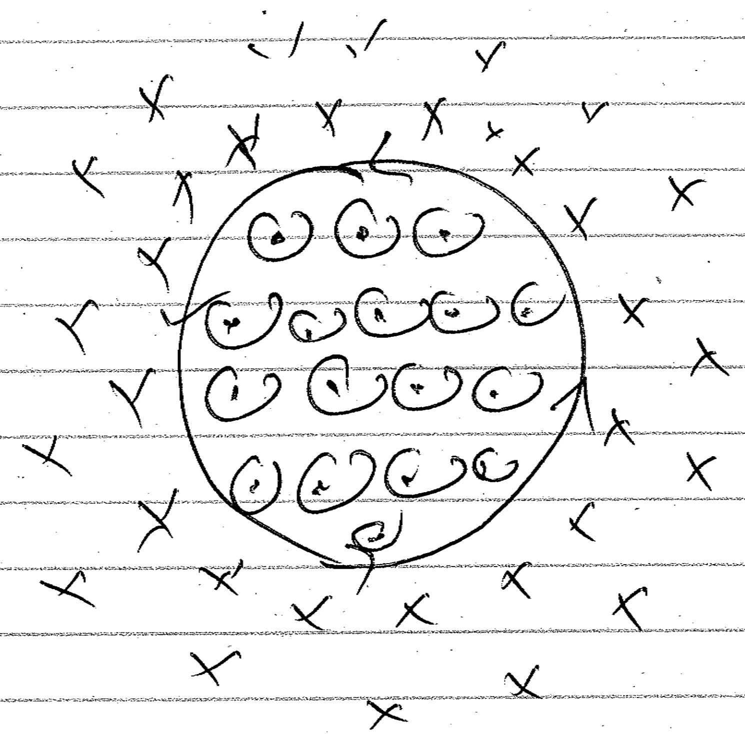
Superposition of $\vec{B}$ from multiple straight current carrying segments

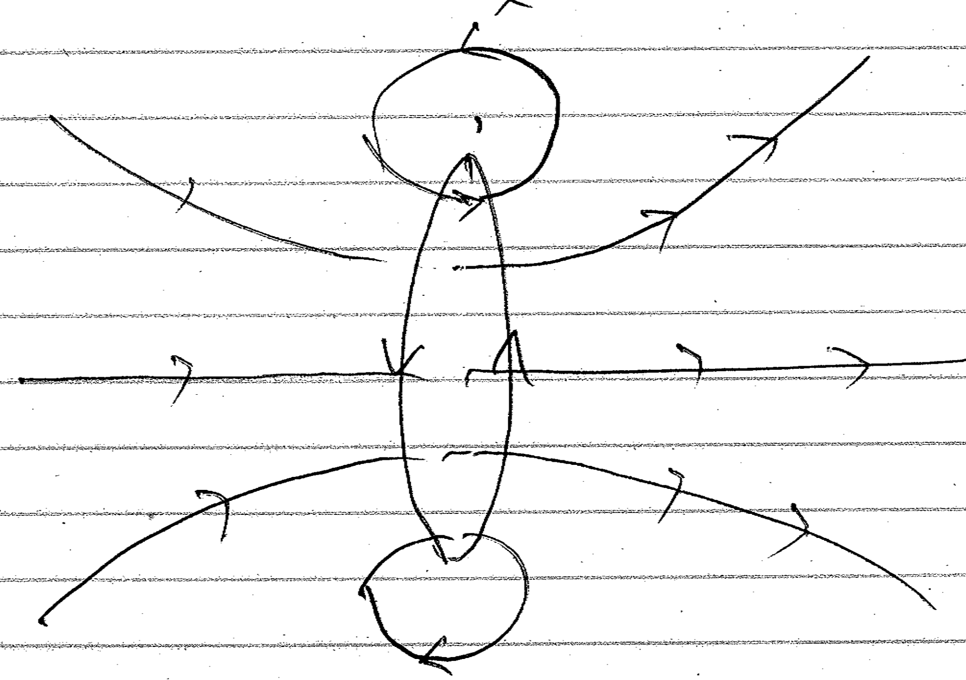
<u>Example #1</u>



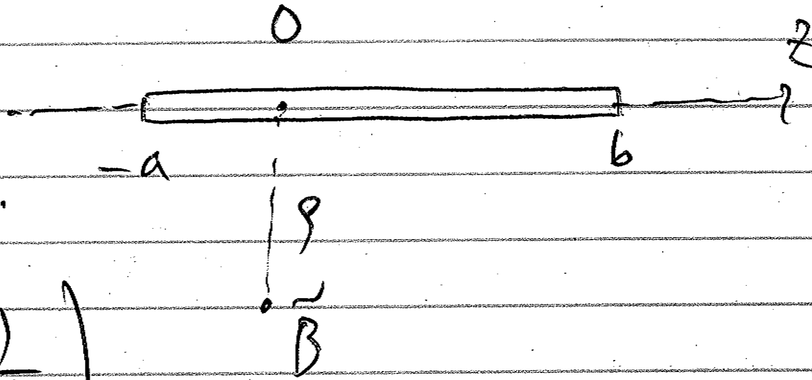


Example #2:



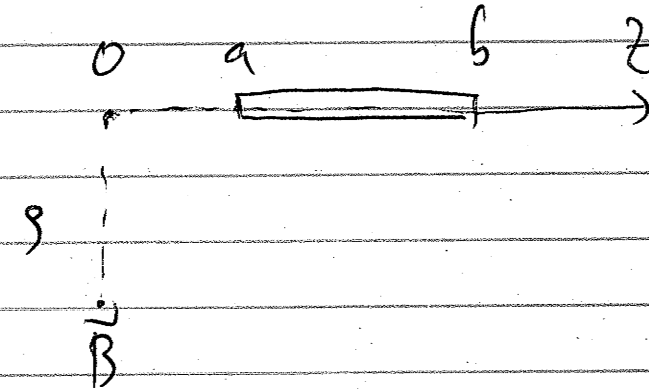


Example #3

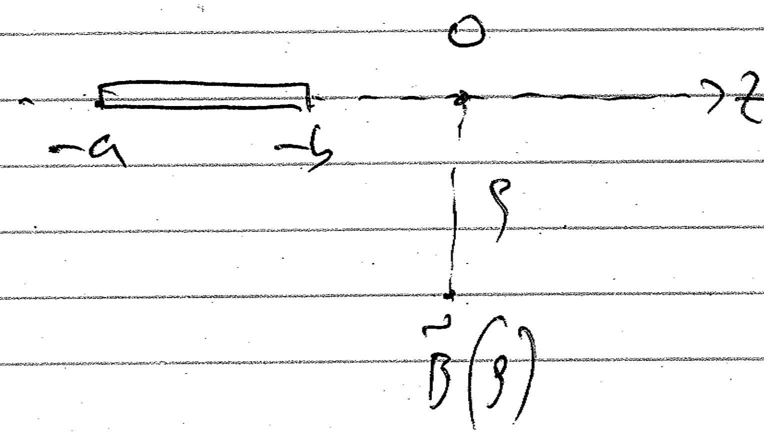


1) $$\vec{B}(\rho) = \hat{e}_\phi \cdot \frac{\mu_0 I}{4\pi \cdot \rho} \cdot \left(\frac{b}{\sqrt{b^2+\rho^2}} - \frac{(-a)}{\sqrt{a^2+\rho^2}} \right)$$

$$= \hat{e}_\phi \frac{\mu_0 I}{4\pi\rho} \left(\frac{b}{\sqrt{b^2+\rho^2}} + \frac{a}{\sqrt{a^2+\rho^2}} \right)$$

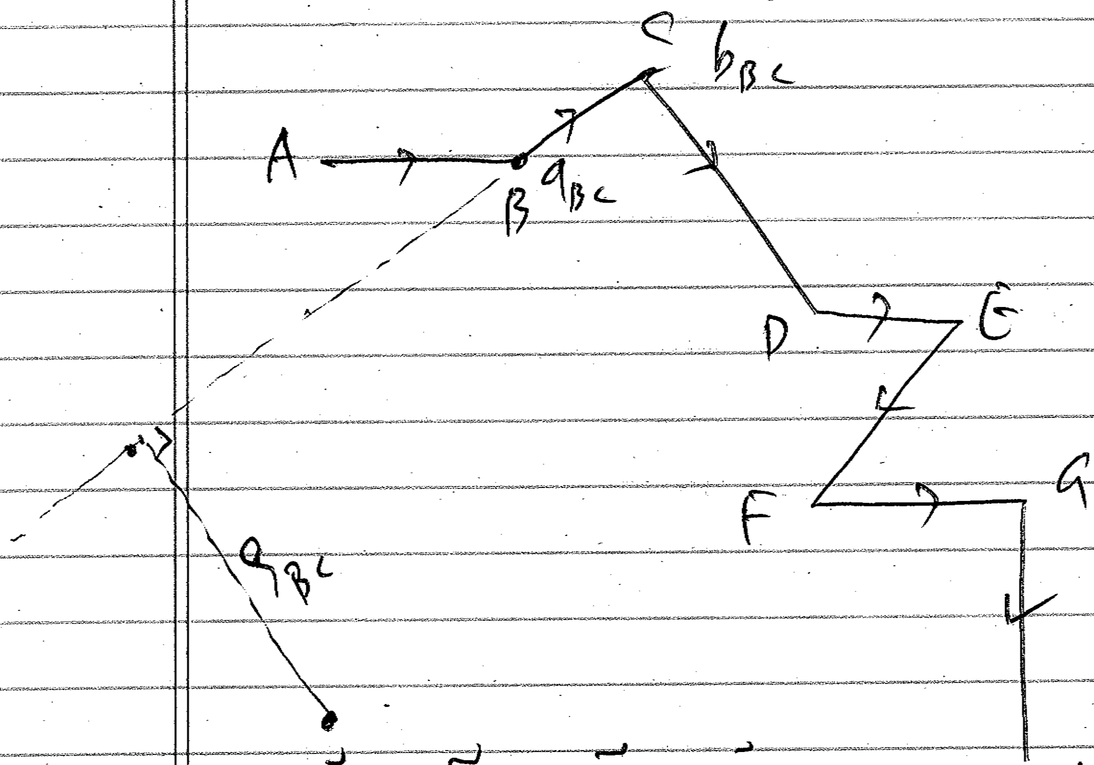


2) $$\vec{B}(\rho) = \hat{e}_\phi \cdot \frac{\mu_0 I}{4\pi \cdot \rho} \cdot \left(\frac{b}{\sqrt{b^2+\rho^2}} - \frac{a}{\sqrt{a^2+\rho^2}} \right)$$



3) $$\vec{B}(\rho) = \hat{e}_\phi \frac{\mu_0 I}{4\pi \cdot \rho} \cdot \left(\frac{-b}{\sqrt{b^2+\rho^2}} - \frac{-a}{\sqrt{a^2+\rho^2}} \right)$$

Superposition of $\vec{B}$ from multiple straight segments

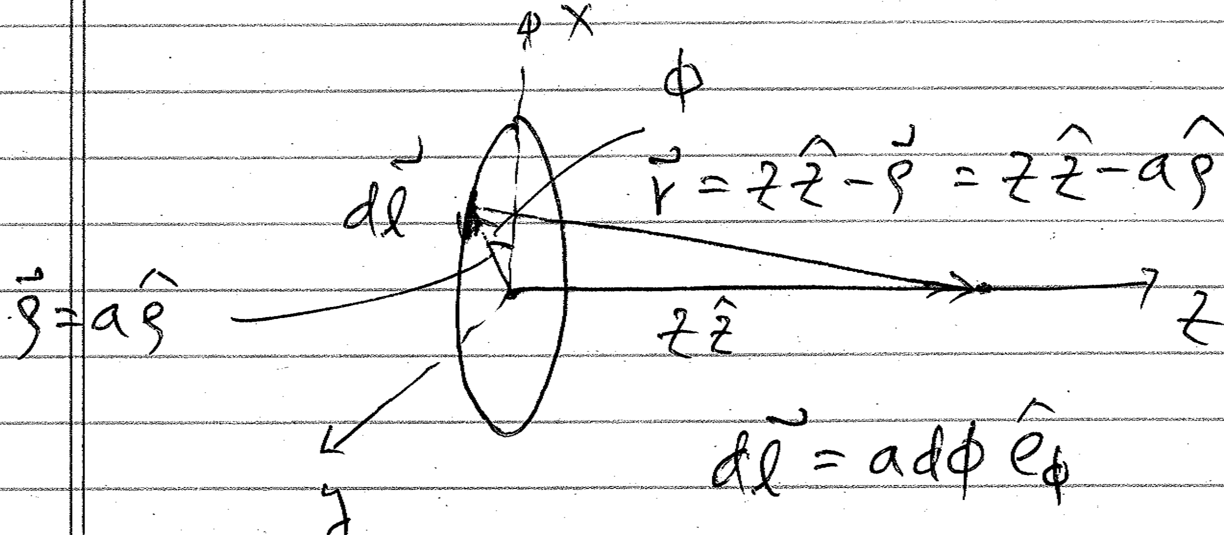


$$\vec{B} = \vec{B}_{AB} + \vec{B}_{BC} + \vec{B}_{CD} + \ldots$$

$$\vec{B}_{BC} = \hat{e}_{\phi,BC} \cdot \frac{\mu_0 I}{4\pi \rho_{BC}} \cdot \left(\frac{b_{BC}}{\sqrt{b_{BC}^2 + \rho_{BC}^2}} - \frac{a_{BC}}{\sqrt{a_{BC}^2 + \rho_{BC}^2}} \right)$$

etc.

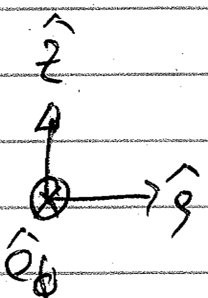
$\vec{B}(z)$ on the axis of a circular current loop



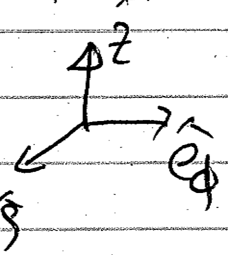
$$d\vec{l} = a\,d\phi\,\hat{e}_\phi$$

$$d\vec{B}\Big|_{d\vec{l}} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{l} \times (z\hat{z} - a\hat{\rho})}{(a^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 \cdot I \cdot d\phi}{4\pi \cdot (a^2+z^2)^{3/2}} \left(az\hat{\rho} + a^2\hat{z}\right)$$



$$\oint \hat{\rho}\, d\phi = 0 \; ;$$



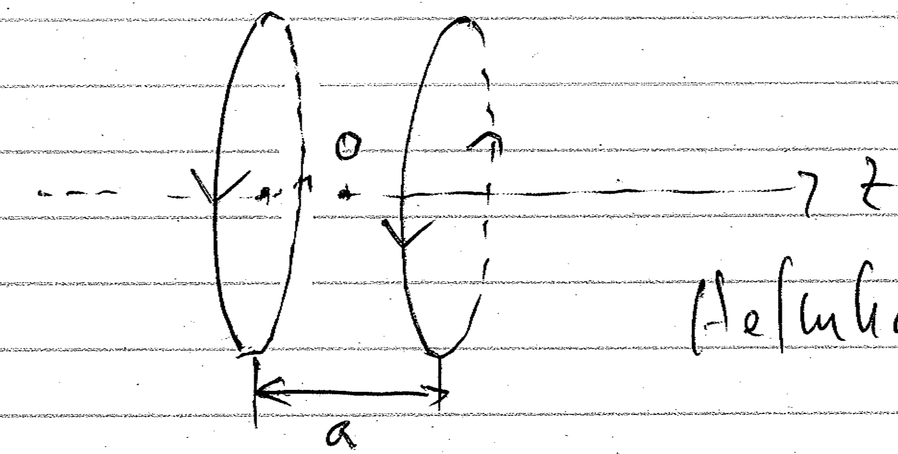
$$\vec{B} = \oint d\vec{B} = \frac{\mu_0 \cdot I \cdot a^2}{2(a^2+z^2)^{3/2}}\hat{z}$$

$\hat{e}_\phi \times \hat{z} = \hat{\rho}$

$\hat{e}_\phi \times \hat{\rho} = -\hat{z}$

$$\vec{B}(z) = \hat{z} \cdot \frac{\mu_0 \cdot I \cdot a^2}{2 \cdot (a^2+z^2)^{3/2}}$$

Production of uniform magnetic field along z-axis with a pair of circular current loops



Helmholtz coil

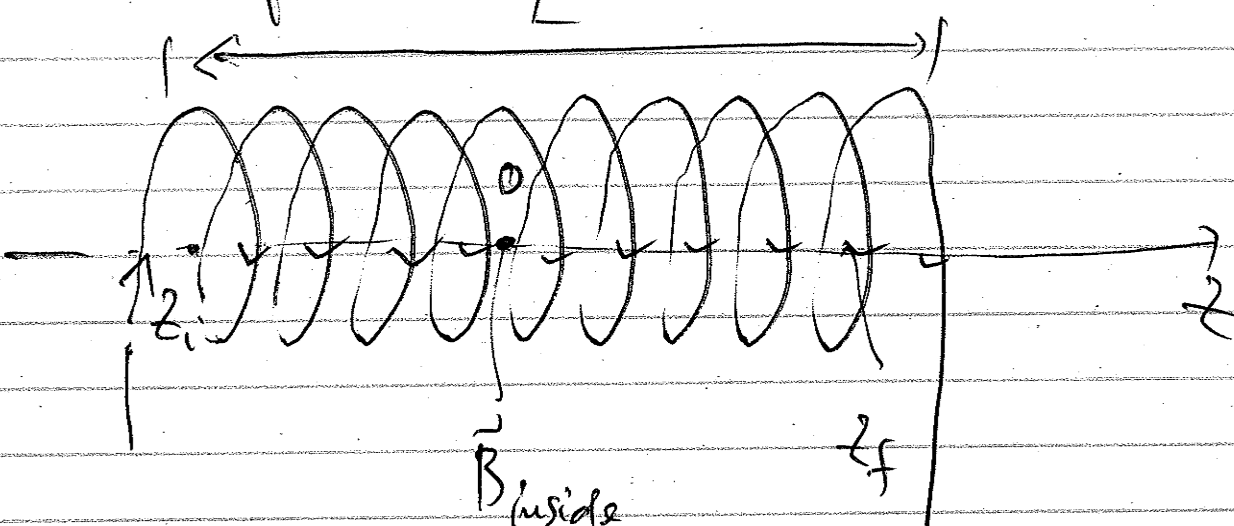
$$\vec{B}(z) = \hat{z}\,\frac{\mu_0 I a^2}{2}\left[\frac{1}{\left(a^2+\left(z+\frac{a}{2}\right)^2\right)^{3/2}}+\frac{1}{\left(a^2+\left(z-\frac{a}{2}\right)^2\right)^{3/2}}\right]$$

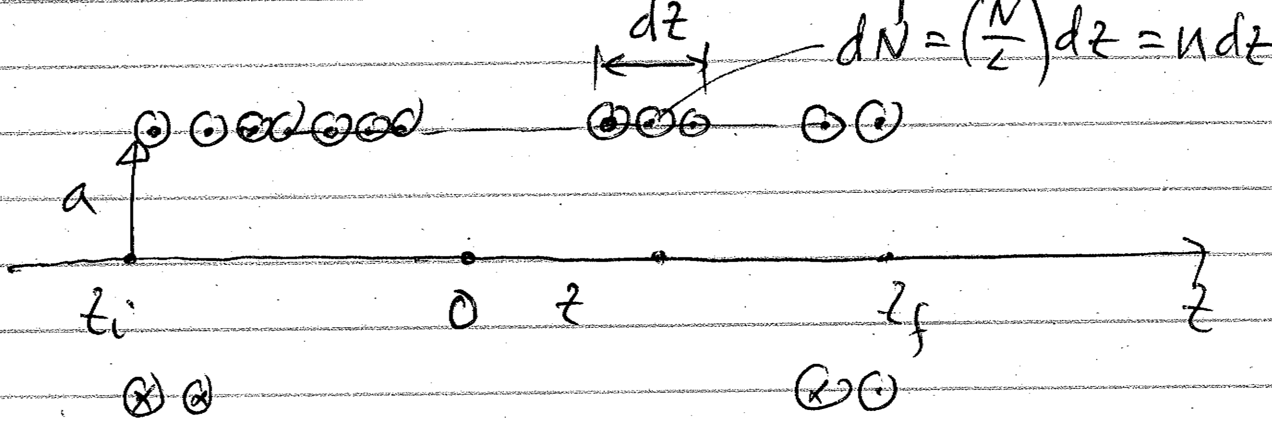
Helmholtz coils

$$\frac{dB}{dz} @ z=0 = 0$$

$$\frac{d^2B}{dz^2} @ z=0 = 0$$

$\vec{B}(z)$ along the axis of a regular array of circular current carrying loops





$$d\vec{B} = \hat{z}\, \frac{\mu_0 I\, dN \cdot a^2}{2} \cdot \frac{1}{(a^2+z^2)^{3/2}}$$

$$\vec{B} = \hat{z}\, \frac{\mu_0 I \cdot n}{2}\, a^2 \int_{z_i}^{z_f} \frac{dz}{(a^2+z^2)^{3/2}}$$

$$= \hat{z}\, \frac{\mu_0 I n}{2} \left(\frac{z_f}{\sqrt{a^2+z_f^2}} - \frac{z_i}{\sqrt{a^2+z_i^2}} \right)$$

Deep inside a long solenoid so that
$z_f \to +\infty, \quad z_i \to -\infty$

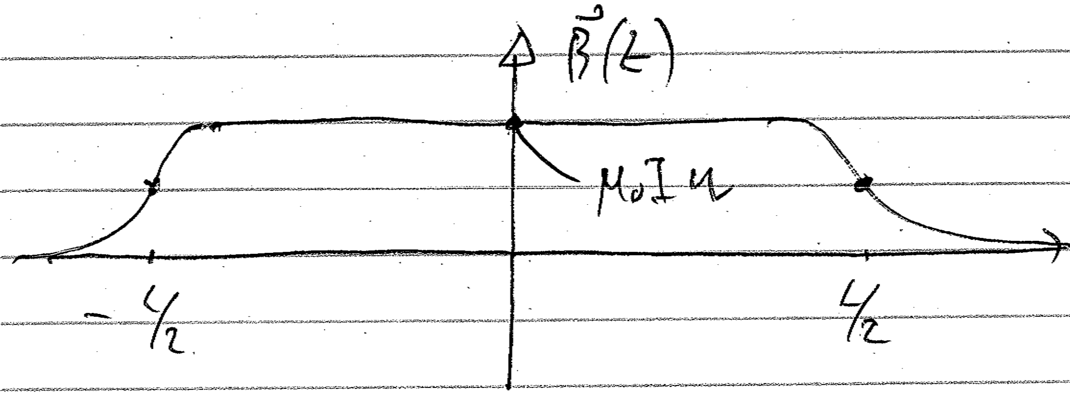
$$\vec{B}_{inside} = \hat{z}\,\mu_0 I n = \hat{z}\,\mu_0 I \left(\frac{N}{L}\right)$$

At either end of a long solenoid,
either $z_f \to +\infty$, $z_i = 0$; or $z_f = 0$, $z_i = -\infty$

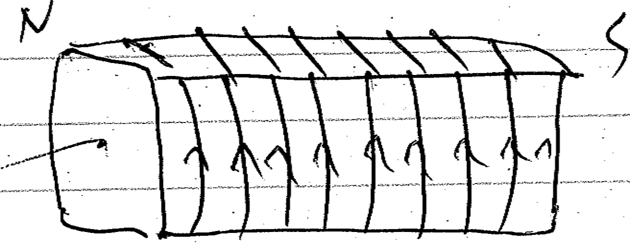
$$\vec{B}_{end} = \hat{z}\,\frac{1}{2}\mu_0 I n = \frac{1}{2}\vec{B}_{inside}$$

Beyond the ends of a long solenoids

$$\vec{B}(z)_{outside} \sim \hat{z}\,\frac{\mu_0 I n}{2} \cdot \frac{a^2}{z^2} \to 0$$

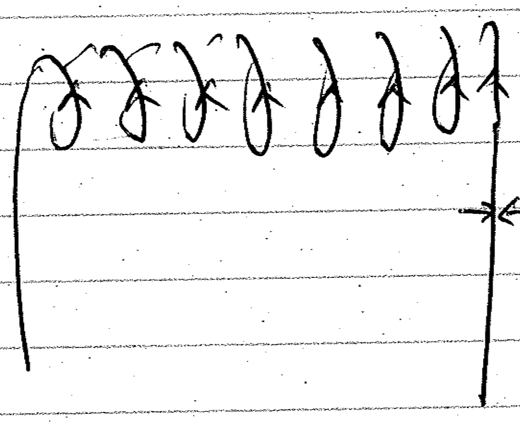


Rectangular permanent magnet



$\tilde{B} = 0.08\,T$

$=$



$d = 2\,mm$

gauge 12

$$\frac{N}{l} = \frac{(l/d)}{l} = \frac{1}{d} = 500\ m^{-1}$$

$$|\tilde{B}|_{edge} = \frac{1}{2}\mu_0 I \left(\frac{N}{l}\right) = 0.08\,T$$

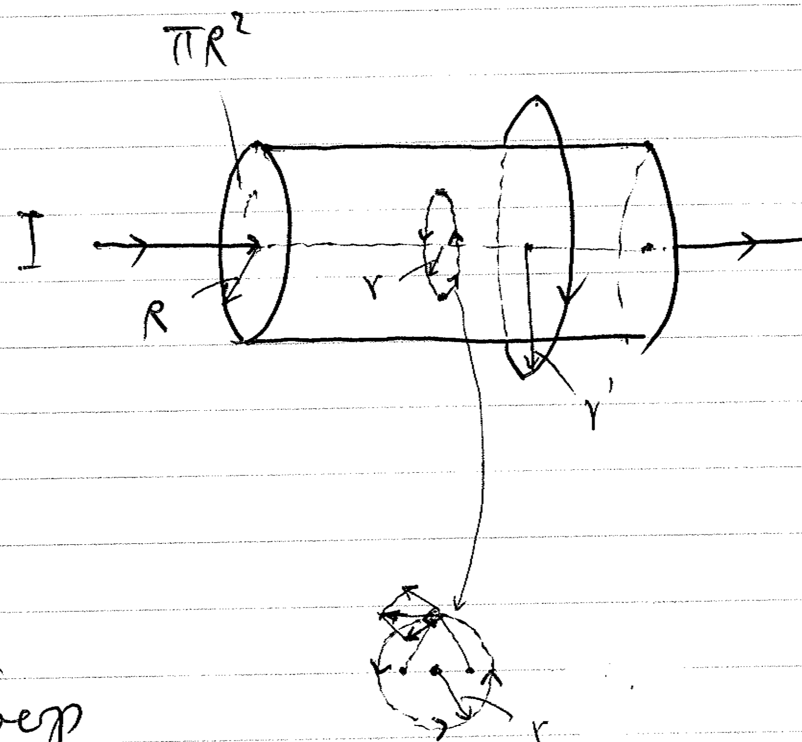
$$\therefore\ I_{equivalent} = \frac{2\,|\tilde{B}|_{edge}}{\mu_0 (N/l)} = 280\ Amp\ !!!$$

Maximum current rating of 12 gauge wire $= 43$ Amp.

If $d = 1\,mm$, $I_{equivalent} = 140$ Amp (Max $= 16$ Amp)

• Magnetic field inside and outside a current-carrying cylindrical wire with a uniform current density

By symmetry, there can be no axial component of B (inside/outside)

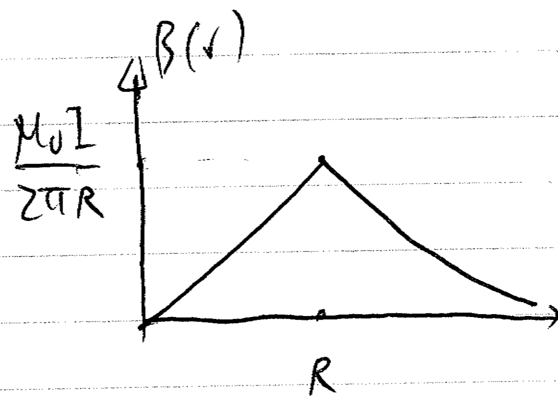


Also by symmetry there can be no radial component of $\vec{B}$ (inside/outside)

Along the smaller loop (inside)

$$\oint_{C_r} \vec{B}\cdot d\vec{\ell} = B(r)\cdot 2\pi r = \mu_0 \left(\frac{\pi r^2}{\pi R^2}\right)\cdot I$$

$$B(r)\Big|_{r<R} = \frac{\mu_0 I}{2\pi R^2}\, r$$

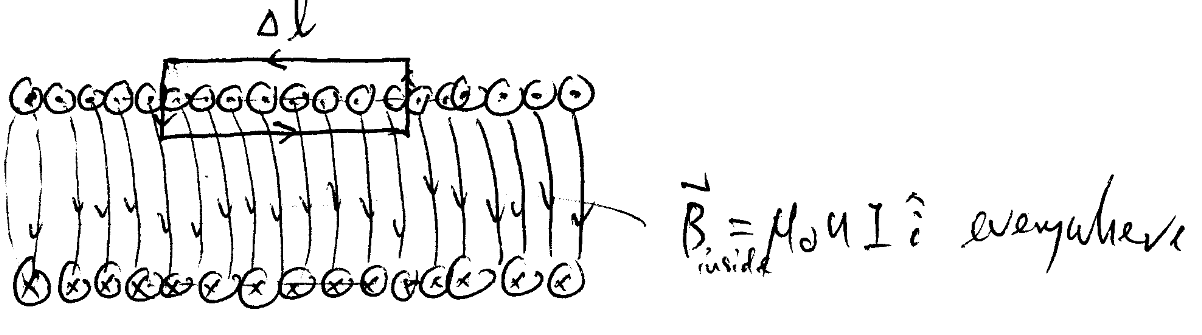


Along the bigger loop,

$$\oint_{C_{r'}} \vec{B}\cdot d\vec{\ell} = B(r')\cdot 2\pi r' = \mu_0 I$$

$$B(r' > R) = \frac{\mu_0 I}{2\pi}\frac{1}{r'}$$

- Magnetic field outside a solenoid or a bar of magnet with S- and N-poles at the ends



$$\oint_c \vec{B}\cdot d\vec{l} = \Delta l \left(B_{inside} - \vec{B}_{outside}\cdot\hat{i}\right)$$

$$= \mu_0 \cdot \Delta l \cdot n \cdot I$$

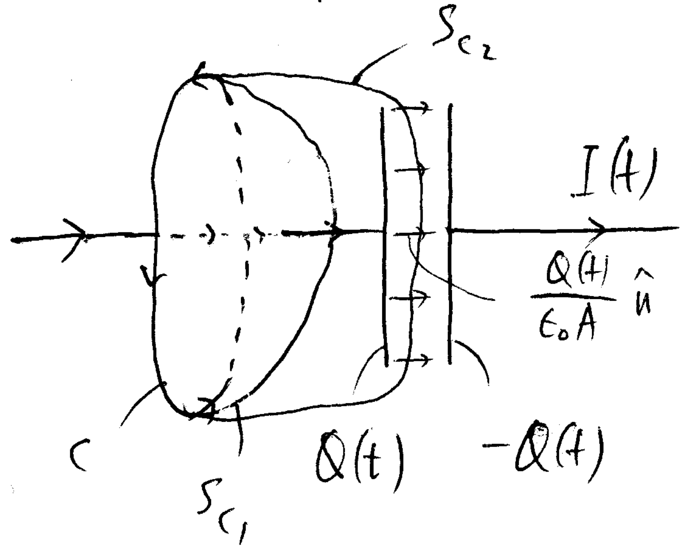
But $B_{inside} = \mu_0 \cdot n \cdot I \Rightarrow \boxed{\vec{B}_{outside}\cdot\hat{i} = 0}$

- Displacement current I_D and Ampere's law:

$$\oint_c \vec{B}\cdot d\vec{l} = \iint_{S_{c_1}} \mu_0 \vec{J}\cdot d\vec{A}$$

$$= \iint_{S_{c_2}} \mu_0 \vec{J}\cdot d\vec{A}$$

But no $\vec{J}$ through S_{c_2}!!



Maxwell added another current to fix this mathematical inconvenience.

$$\vec{J}_D = \epsilon_0 \frac{d\vec{E}}{dt} \quad \text{or} \quad \kappa\epsilon_0 \frac{d\vec{E}}{dt} \quad \text{(in dielectric)}$$

$$I_D = \iint_{S_c} \vec{J}_D \cdot d\vec{A} = \kappa\epsilon_0 \frac{d}{dt} \underbrace{\iint_{S_c} \vec{E} \cdot d\vec{A}}_{\Phi_E}$$

Inside capacitor, $\vec{J} = 0$, but $\vec{J}_D = \epsilon_0 \frac{d}{dt} \vec{E}$,

$$I_D = \iint_{S_{c2}} \vec{J}_D \cdot d\vec{A} = \iint_{S_{c2}} \epsilon_0 \frac{d}{dt}\left(\frac{Q}{\epsilon_0 A}\hat{n}\right) \cdot \hat{n}\, dA = \frac{dQ}{dt}$$

So, generally

$$\boxed{\oint_c \vec{B} \cdot d\vec{\ell} = \mu_0 \left(I_{free} + I_D\right)_{encl.}}$$

Derivation of Ampere's law and Maxwell's Displacement current from Biot-Savart law

$$\vec{B}(\vec{r}) = \left(\frac{\mu_0}{4\pi}\right) \iiint d\vec{r}\,' \, \vec{j}\,' \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

(Biot-Savart law)

Now,

$$\oint_c \vec{B}\cdot d\vec{\ell} = \iint_{S_c} (\nabla\times\vec{B})\cdot d\vec{s}$$

$$= \left(\frac{\mu_0}{4\pi}\right) \iint_{S_c} d\vec{s}\cdot \iiint d\vec{r}\,' \, \nabla\times\left(\vec{j}\,' \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right)$$

$$= \left(\frac{\mu_0}{4\pi}\right) \iint_{S_c} d\vec{s}\cdot \iiint d\vec{r}\,' \, \nabla\times\left(\nabla\frac{1}{|\vec{r}-\vec{r}\,'|}\times\vec{j}\,'\right)$$

from $\nabla\times(\vec{a}\times\vec{b}) = \vec{a}(\nabla\cdot\vec{b}) - \vec{b}(\nabla\cdot\vec{a}) + (\vec{b}\cdot\nabla)\vec{a} - (\vec{a}\cdot\nabla)\vec{b}$

$$\nabla\times\left(\nabla\frac{1}{|\vec{r}-\vec{r}\,'|}\times\vec{j}\,'\right) = -\vec{j}\,'\nabla^2\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) + (\vec{j}\,'\cdot\nabla)\nabla\frac{1}{|\vec{r}-\vec{r}\,'|}$$

(∇ only acts on $\vec{r}$, not on $\vec{r}\,'$)

Integrate the second term by part and assuming that $\vec{j}'$ vanishes at infinity

$$\iiint d\vec{r}' \, (\vec{j}' \cdot \nabla) \nabla \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \nabla \iiint d\vec{r}' \, (\vec{j}' \cdot \nabla) \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= -\nabla \iiint d\vec{r}' \, (\vec{j}' \cdot \nabla') \frac{1}{|\vec{r}-\vec{r}'|} \qquad (\nabla' \text{ acts on } \vec{r}' \text{ only})$$

$$= \nabla \iiint d\vec{r}' \, \frac{\nabla' \cdot \vec{j}'}{|\vec{r}-\vec{r}'|}$$

From the charge conservation

$$\nabla' \cdot \vec{j}' + \frac{\partial \rho_f'}{\partial t} = 0$$

We have

$$\iiint d\vec{r}' \, (\vec{j}' \cdot \nabla) \nabla \frac{1}{|\vec{r}-\vec{r}'|} = \frac{d}{dt} \left(- \nabla \iiint d\vec{r}' \frac{\rho_f'}{|\vec{r}-\vec{r}'|} \right)$$

$$= (4\pi\epsilon_0) \frac{d}{dt} \vec{D}$$

Finally

$$\oint_C \vec{B}\cdot d\vec{l} = \left(\frac{\mu_0}{4\pi}\right) \iint_{S_c} d\vec{s}\cdot \left\{ - \iiint d\vec{r}\,' \vec{j}\,' \nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}\,'|} \right) \right.$$

$$\left. + 4\pi\epsilon_0 \frac{d\vec{D}}{dt} \right\}$$

$$-\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\, \delta(\vec{r}-\vec{r}\,')$$

so

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \iint_{S_c} d\vec{s}\cdot\vec{j} + \mu_0\epsilon_0 \frac{d}{dt} \iint_{S_c} \vec{D}\cdot d\vec{s}$$

or

$$\boxed{\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\epsilon_0 \frac{d}{dt} \iint_{S_c} \vec{D}\cdot d\vec{s}}$$

In differential equation form,

$$\boxed{\nabla\times\vec{B} = \mu_0 \vec{j} + \mu_0\epsilon_0 \frac{\partial\vec{D}}{\partial t}}$$

⁂

Proof that $\nabla \cdot \vec{B} = 0$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint d\sigma' \, \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

$$\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint dv' \, \nabla \cdot \left(\vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)$$

Since $\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot (\nabla \times \vec{b})$
and ∇ only acts on $\vec{r}$, then

$$\nabla \cdot \left(\vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)$$

$$= - \vec{J}(\vec{r}') \cdot \left(\nabla \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right)$$

Since $\nabla \times (\psi \vec{a}) = \nabla \psi \times \vec{a} + \psi \nabla \times \vec{a}$, with

$\psi = \frac{1}{|\vec{r} - \vec{r}'|^3}$, $\quad a = \vec{r} - \vec{r}'$

$$\nabla \times (\psi \vec{a}) = \nabla \left(\frac{1}{|\vec{r} - \vec{r}'|^3} \right) \times (\vec{r} - \vec{r}') + \frac{1}{|\vec{r} - \vec{r}'|^3} \left(\nabla \times (\vec{r} - \vec{r}') \right)$$

$$= 0$$

Thus

$$\nabla\cdot\vec{B} = 0$$

This can also shew classically as follows:

Since each current segment $dv'\vec{J}(\vec{r}')$ produces ~~which~~ circulating magnetic fields about $\hat{J}(\vec{r}')$, these magnetic flux either never cut through a closed surface, or exit as many times through a closed surface as enter. So the net flux through a close surface is zero.

By the principle of superposition, the magnetic field lines (flux) as described by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint dv'\vec{J}(\vec{r}') \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

through a close surface is always zero.

$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$

Chapt. 29

- Magnetic induction (Faraday's law) (Week 10)

$$
\begin{cases}
\oint\!\!\oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{in}}{\epsilon_0} \\[2ex]
\oint_C \vec{E}\cdot d\vec{l} = 0 \implies -\dfrac{d}{dt}\iint \vec{B}\cdot d\vec{A} \quad \text{(Faraday's law)} \\[2ex]
\oint\!\!\oint \vec{B}\cdot d\vec{A} = 0 \\[2ex]
\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \kappa \underbrace{\mu_0 \epsilon_0}_{=\frac{1}{c^2}} \dfrac{d}{dt}\iint \vec{E}\cdot d\vec{A}
\end{cases}
$$

$$\implies \frac{\kappa}{c^2} = \frac{1}{v^2}$$

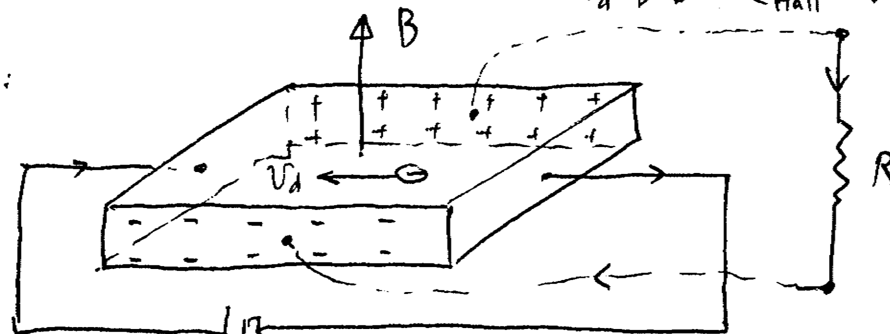
$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad v = \frac{c}{n} = \frac{c}{\sqrt{\kappa}}$$

- Electromotive forces produced by moving charges in a steady magnetic field

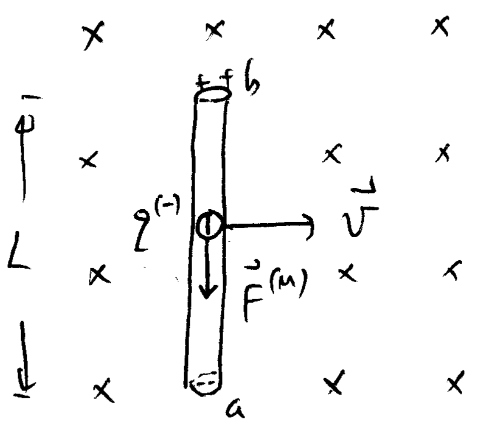
$V_d \cdot B \cdot b = \mathcal{E}_{Hall} = V_{Hall}$

Hall Effect:

(Special case of motion-emf)



Moving conducting rod in a uniform magnetic field



$\vec{B}$ (into the board)

$$\mathcal{E} = \int_a^b \frac{\vec{F}^{(M)}(\text{on } q^{(-)})}{q^{(-)}} \cdot d\vec{\ell}$$

$d\vec{\ell} \parallel \vec{v} \times \vec{B}$

$$= \int_a^b \frac{q^{(-)}(\vec{v} \times \vec{B}) \cdot d\vec{\ell}}{q^{(-)}}$$

$$= v \cdot B \cdot L$$

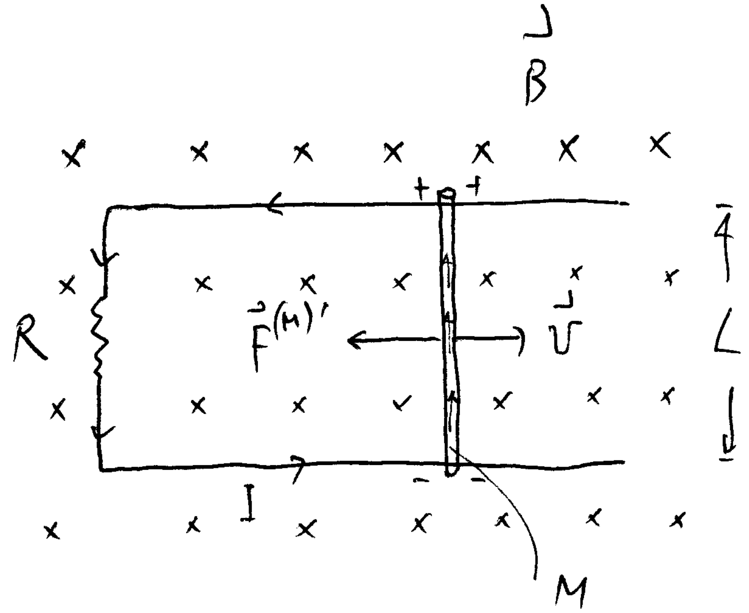
$$\boxed{\mathcal{E}_m = v \cdot B \cdot L}$$

Example 30-7

A rod slides frictionlessly to right with a velocity $\vec{v}$

$\mathcal{E} = v \cdot B \cdot L$

$$I = \frac{\mathcal{E}}{R} = \frac{vBL}{R}$$



Additional magnetic force on the rod

$$\vec{F}^{(M)\prime} = I\vec{L} \times \vec{B}$$

decelerates the rod so that

$$M\frac{dv}{dt} = -ILB = -\frac{B^2L^2}{R}v$$

$$v(t) = v_0 e^{-(B^2L^2/MR)\cdot t} = v_0 e^{-t/\tau_c}$$

With $B = 0.2\,T$, $L = 0.1\,m$, $M = 0.01\,Kg$, $R = 0.1\,\Omega$

$$\tau_c = \frac{MR}{(BL)^2} = \frac{0.01 \times 0.1}{(0.2)^2(0.1)^2} = 2.5\ s.$$

Power dissipation in R:

$$P = I^2R = \frac{\mathcal{E}^2}{R} = \frac{B^2L^2}{R}v^2 = \frac{B^2L^2}{R}v_0^2 e^{-2t/\tau_c}$$

$$U = \int_0^\infty P\,dt = \frac{M}{2}v_0^2$$ (all the initial kinetic energy).

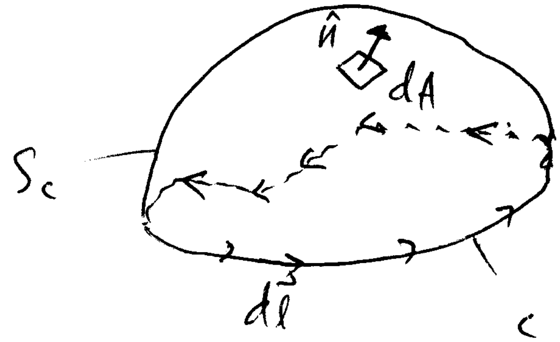
$\boxed{\text{Example 30-11, read it yourself}}$ $\Rightarrow$ Eddy current demonstration

- Electromotive force produced by changing magnetic field

<u>Faraday's law</u>:

A change in the magnetic flux through a close loop with a designated direction induces a net emf along the loop, and the emf equals to the negative of the time rate of the flux change

$$\Phi_B = \iint_{S_c} \vec{B}\cdot d\vec{A}$$



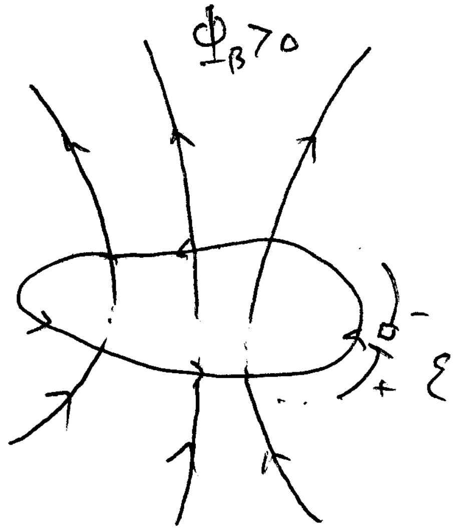
With $\hat{n}$ of $d\vec{A} = dA\,\hat{n}$ chosen to obey Right hand rule with the designated loop direction

$$\boxed{\mathcal{E}\ (\text{along the loop direction}) = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A}}$$

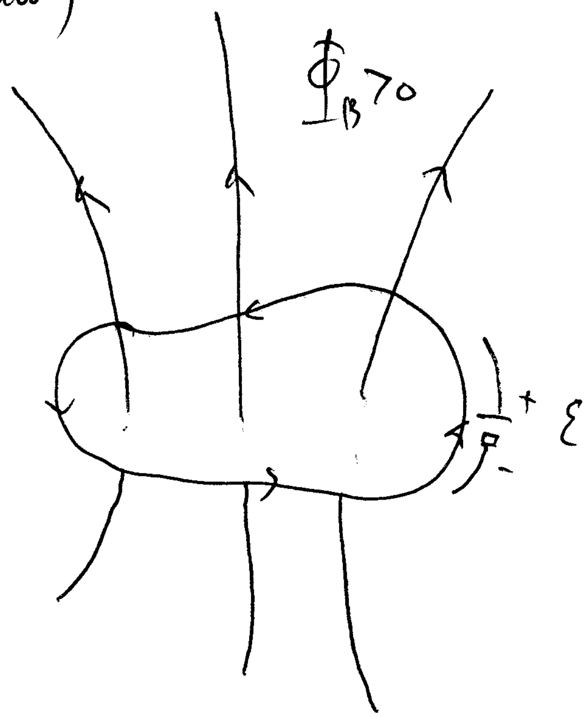
Since the induced emf would produce a current that, in turn generate a magnetic flux to compensate the "loss" of the original magnetic flux

<u>Lenz' law</u>: The direction of a magnetic induced emf is such as to oppose the cause of the induction.

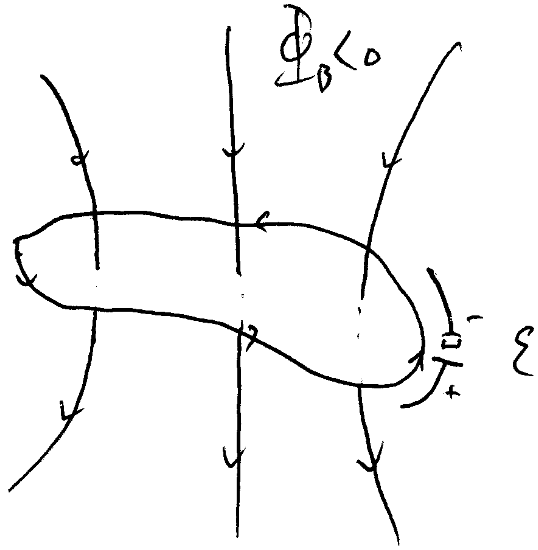
Direction of emf: (Lenz' law)



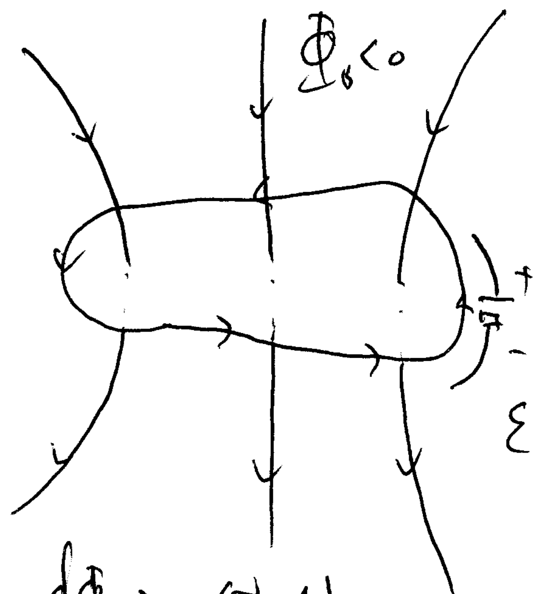
$\frac{d\Phi_B}{dt} \uparrow$, $\mathcal{E} < 0$



$\frac{d\Phi_B}{dt} \downarrow$, $\mathcal{E} > 0$

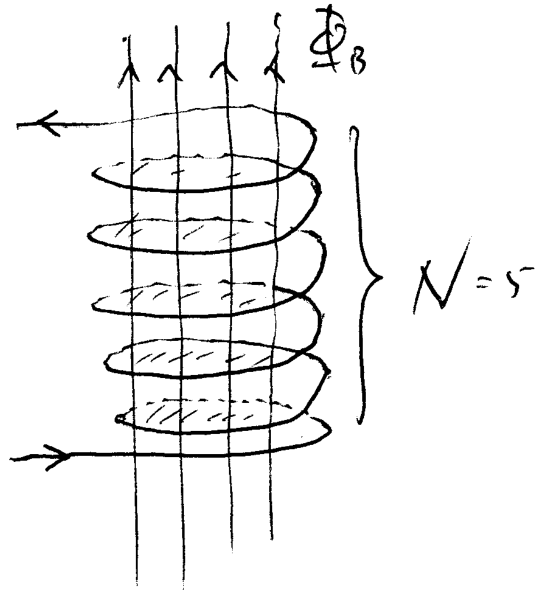


$\frac{d\Phi_B}{dt} \uparrow$ ($\vec{B} \downarrow$), $\mathcal{E} < 0$



$\frac{d\Phi_B}{dt} \downarrow$ ($\vec{B} \uparrow$), $\mathcal{E} > 0$

Magnetic induction through a close loop with N identical turns



Along each turn,

$$\Delta \mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\mathcal{E} = N \cdot \Delta\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

$\boxed{\text{Example 30-1}}$

$\boxed{\text{Example 30-3}}$

- Induced electric field by changing magnetic field.

What is the driving force $\vec{F}_n$ that leads to $\mathcal{E} = \int \frac{\vec{F}_n}{q} \cdot d\vec{l}$?

There is nothing else in space, can't be chemical force such as what is in a dry battery

Nothing is moving in a close loop (yet), can't be a magnetic force (between moving charges)

It is a circulating electric field $\vec{E}$:

$$\vec{F}_n = q\vec{E}$$

$$\mathcal{E} = \oint (\vec{F}_n \cdot d\vec{l})/q = \oint \vec{E} \cdot d\vec{l}$$

$$\boxed{\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint \vec{B} \cdot d\vec{A}} \quad \text{(Faraday's law)}$$

This "magnetically induced" electric field is non-conservative, or a transverse field such that

$$\oint \vec{E}_{non} \cdot d\vec{l} = \oint \vec{E}_T \cdot d\vec{l} \neq 0$$

$$\nabla \times \vec{E}_{non} = \nabla \times \vec{E}_T \neq 0$$

It is to be distinguished from the electro-static field $\vec{E}_c$ produced by electric charges (longitudinal field)

$$\oint \vec{E}_L \cdot d\vec{l} = 0$$

Generally, $\vec{E} = \vec{E}_c + \vec{E}_N = \vec{E}_L + \vec{E}_T$ * (Maxwell's equations)

$\boxed{\text{Example 30-12}}$

General Proof of Motion EMF being equivalent to Faraday's law without any induced transverse electric field $\vec{E}_T$

—— EMF induced by relative motion of a magnet and a conducting loop (wire) can be equally well described by

$$\mathcal{E} = -\frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A} \qquad \text{(with care)}$$

and

$$\mathcal{E} = \oint_c d\vec{l}\cdot(\vec{v}\times\vec{B})$$

In the absence of mobile charges, motion emf is zero, amounting to nothing.

Yet time change in magnetic field $\vec{B}$ in the absence of any motion always produces a transverse electric field $\vec{E}_T$ so that

$$\oint_c \vec{E}_T\cdot d\vec{l} = -\frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A}$$

The key is that the motion emf

$$\mathcal{E} = \oint d\vec{l} \cdot (\vec{v} \times \vec{B})$$

can be computed through

$$\oiint_S \vec{B} \cdot d\vec{s} = 0$$

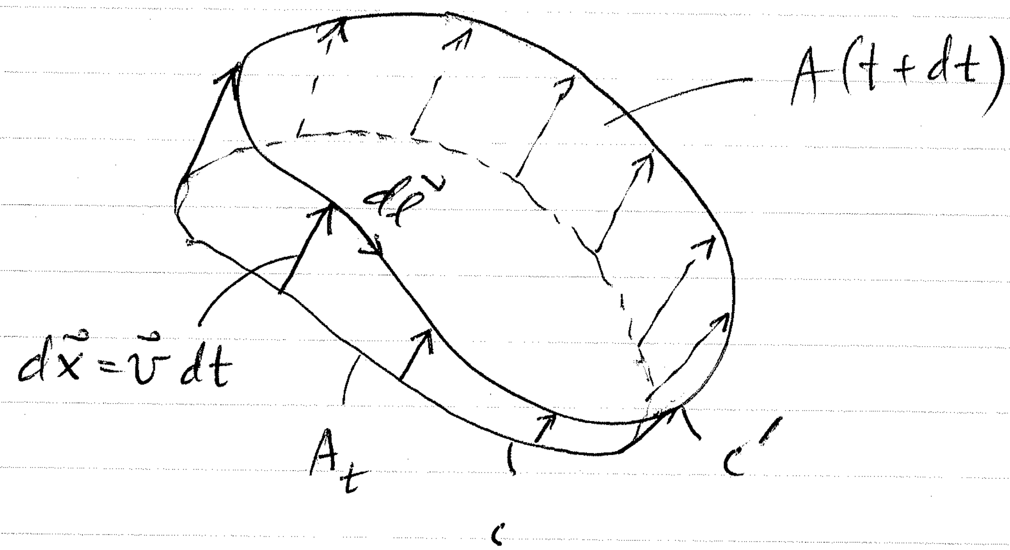
using a generalized Faraday's induction law

$$\mathcal{E}_{total} = -\frac{d}{dt} \iint_{S_c} \vec{B} \cdot d\vec{s}$$

$$= -\iint_{S_c \text{ (fixed)}} \left(\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{s}$$

$$- \frac{d}{dt} \iint_{S_c \text{ (varying)}} \vec{B}(\text{fixed}) \cdot d\vec{s}$$

Let c be a closed wire loop at an instant t; at a later time, the same wire loop moves to a new position c'.



The magnetic flux leaving the box as shown is zero

$$\oiint_S \vec{B}\cdot d\vec{A} = 0 = \iint_{A(t)} \vec{B}\cdot d\vec{A} + \iint_{A(t+dt)} \vec{B}\cdot d\vec{A}$$

$$+ \oint_c \vec{B}\cdot\left(d\vec{\ell} \times d\vec{x}\right)$$

$$\iint_{A(t)} \vec{B}\cdot d\vec{A} = -\iint_{S_c} \vec{B}\cdot d\vec{A}\Big|_{at\ t} \quad ; \quad \iint_{A(t+dt)} \vec{B}\cdot d\vec{A} = \iint_{S_c} \vec{B}\cdot d\vec{A}\Big|_{t+dt}$$

Thus

$$\iint_{A(t)} \vec{B}\cdot d\vec{A} + \iint_{A(t+dt)} \vec{B}\cdot d\vec{A} = dt\left(\frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A}\right)$$

But

$$\oint_c \vec{B}\cdot\left(d\vec{l}\times d\vec{x}\right) = dt\oint_c \vec{B}\cdot\left(d\vec{l}\times\vec{v}\right)$$

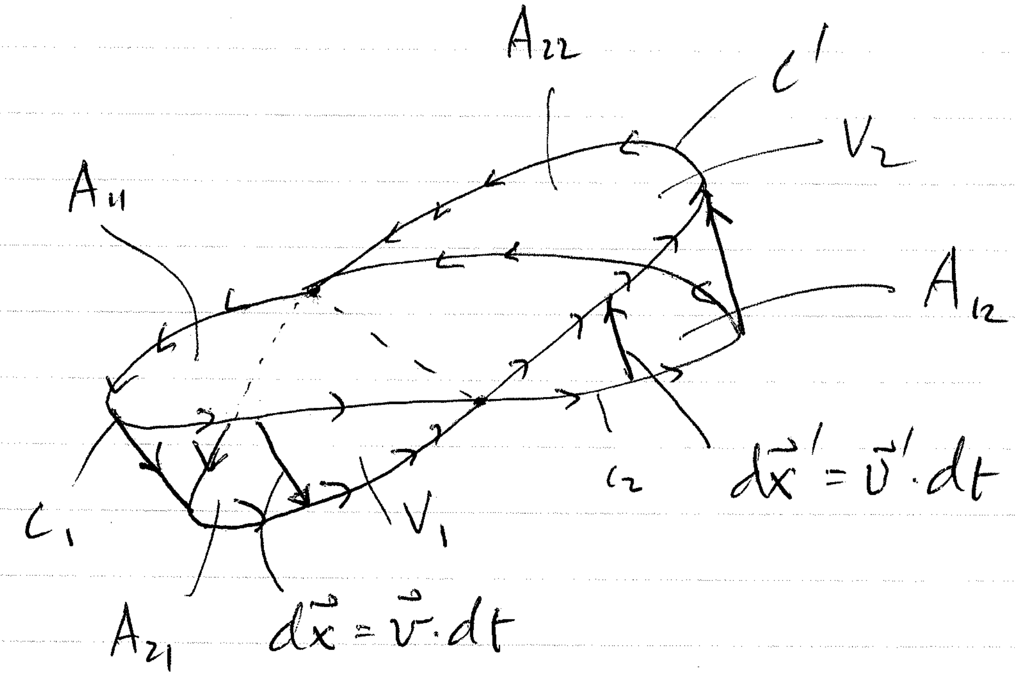
$$= dt\oint_c \left(d\vec{l}\times\vec{v}\right)\cdot\vec{B}$$

$$= dt\oint_c d\vec{l}\cdot\left(\vec{v}\times\vec{B}\right)$$

$$\therefore \frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A} + \oint_c d\vec{l}\cdot\left(\vec{v}\times\vec{B}\right) = 0$$

$$\therefore \oint_c d\vec{l}\cdot\left(\vec{v}\times\vec{B}\right) = -\frac{d}{dt}\iint_{S_c} \vec{B}\cdot d\vec{A}$$

It is possible that two "halves" of a close wire loop move in "opposite" directions as in form of rotation about two points on the loop. We then have the following situation:



Apply Gauss' law to each volume

$$\oiint_{V_1} \vec{B} \cdot d\vec{s} = 0 = \iint_{A_{11}(t)} \vec{B} \cdot d\vec{s} - \iint_{A_{21}(t+dt)} \vec{B} \cdot d\vec{s}$$

$$+ \int_{C_1} \vec{B} \cdot (\vec{v} \times d\vec{l})\, dt = 0 \quad \ldots (1)$$

$$\iint_{V_2} \vec{B}\cdot d\vec{s} = 0 = \iint_{A_{22}(t+dt)} \vec{B}\cdot d\vec{s} - \iint_{A_{12}(t)} \vec{B}\cdot d\vec{s}$$

$$+ \int_{c_2} \vec{B}\cdot(d\vec{l}\times\vec{v})\cdot dt \quad \cdots \text{②}$$

$-$ ① $+$ ② :

$$0 = \iint_{S_c'(t+dt)} \vec{B}\cdot d\vec{s} - \iint_{S_c(t)} \vec{B}\cdot d\vec{s} + \oint_c \vec{B}\cdot(d\vec{l}\times\vec{v})\,dt$$

$$\therefore\ dt\left(-\frac{d}{dt}\iint_{S_c}\vec{B}\cdot d\vec{s}\right) = \left(\oint_c d\vec{l}\cdot(\vec{v}\times\vec{B})\right)\cdot dt$$

$$\boxed{\therefore\ \oint_c d\vec{l}\cdot(\vec{v}\times\vec{B}) = -\frac{d}{dt}\iint_{S_c}\vec{B}\cdot d\vec{A}}$$

☆

The total induced emf

$$\mathcal{E} = \oint_{c_{moving}} d\vec{l} \cdot (\vec{E} + \vec{v} \times \vec{B})$$

$$= - \iint_{S_{c\,fixed}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

$$- \frac{d}{dt} \iint_{S_c\ moving} \vec{B}(\text{fixed}) \cdot d\vec{A}$$

By special relativity, if for each line segment $d\vec{l}$ with a velocity $\vec{v}$ relative to the static magnetic field $\vec{B}$, then in the frame of $d\vec{l}$, the line segment (static) (x'-y'-z' frame) feels a static magnetic field $\vec{B}' \cong \vec{B}$, but also an electric field $\vec{E} = \vec{v} \times \vec{B}$. $\vec{E}$ acts as the source of emf.

$$\mathcal{E} = \oint_c \vec{E} \cdot d\vec{l} = \oint_c d\vec{l} \cdot (\vec{v} \times \vec{B})$$ #

Another way to look at it is

$$\mathcal{E} = \oint_c d\vec{l} \cdot \left(\vec{E}_T + \vec{E}_{relativity} \right)$$

$$\oint_c d\vec{l} \cdot \vec{E}_T = -\iint_{S_c} \left(\frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{s}$$

$$\vec{E}_{relativity} = \vec{v} \times \vec{B}$$

General Faraday's Induction Law

$$\mathcal{E} = \oint_C d\vec{l} \cdot \left(\vec{E}_T + \vec{U} \times \vec{B}\right) = -\frac{d}{dt} \iint_{S_C} \vec{B} \cdot d\vec{A}$$

$$= -\left(\frac{d}{dt} \iint_{S_C \text{ moving}} \vec{B}(\text{fixed}) \cdot d\vec{A} + \iint_{S_C \text{ fixed}} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}\right)$$

If there are N identical turns coils coincide with the loop C, then each turn produces the same emf that connects in series with the rest N-1 turns.

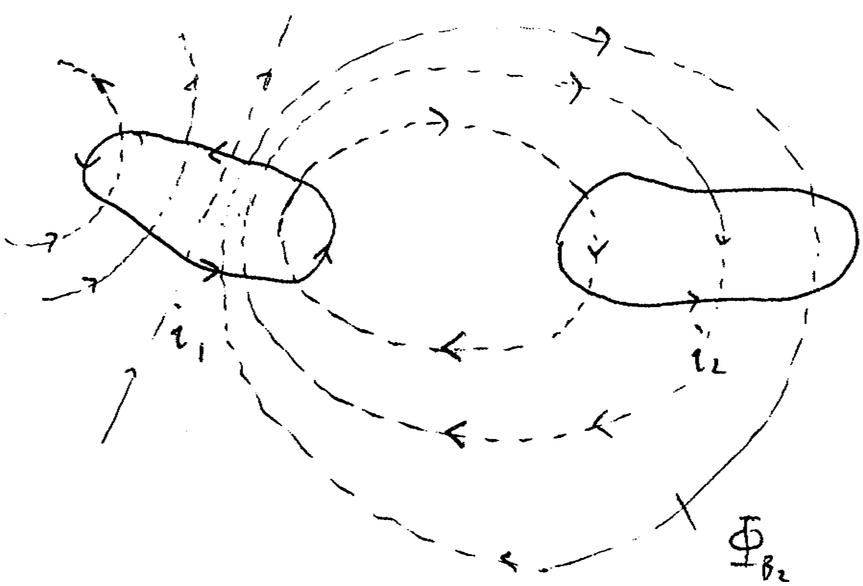
$$\mathcal{E}(N \text{ turns}) = N \oint_C d\vec{l} \cdot \left(\vec{E}_T + \vec{U} \times \vec{B}\right)$$

$$= -N \frac{d}{dt} \iint_{S_C} \vec{B} \cdot d\vec{A}$$

$$= -\frac{d}{dt}\left(N \Phi_B\right)$$

Chapt. 30

- Mutual Inductance and self inductance



A current in loop 1 produces a net magnetic flux through a second loop Φ_{B_2} (proportional to i_1). A time change in i_1 causes Φ_{B_2} to change accordingly, thus induces an emf in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_{B_2}}{dt} = -M_{21}\frac{di_1}{dt}$$

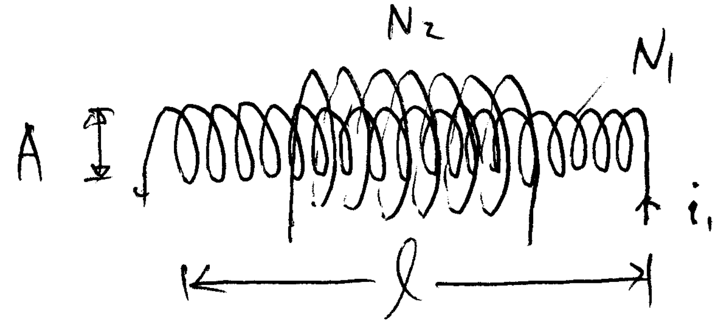
M_{21} : mutual inductance. (S.I. unit: henry, $1H = W/A$)

Similarly, a time change in i_2 in loop 2 causes the magnetic flux Φ_{B_1} (produced by i_2) to change, leading to an emf in loop 1

$$\mathcal{E}_1 = -\frac{d\Phi_{B_1}}{dt} = -M_{12}\frac{di_2}{dt}$$

$$\boxed{M_{21} = M_{12}}$$

Example 31-1. Mutual inductance between two solenoids



$$\Phi_{B_2} = A \cdot \mu_0 \left(\frac{N_1}{l}\right) \cdot i_1$$

$$\mathcal{E}_2 = -\frac{d\Phi_{B_2}}{dt} \cdot N_2$$

$$= -\frac{\mu_0 \cdot A \cdot N_1 \cdot N_2}{l} \cdot \frac{di_1}{dt}$$

$$\therefore \quad M_{12} = M_{21} = M = \frac{\mu_0 A \cdot N_1 \cdot N_2}{l} \quad \text{(function of geometry)}$$

With $l = 0.5\,m$, $A = 10\,cm^2$, $N_1 = 1000$, $N_2 = 10$, $M = 20\,\mu H$

With $i_2 = 2\times10^6\,(A/s)\cdot t$, $di_2/dt = 2\times10^6\,A/s$, $\mathcal{E}_1 = -50\,V$ #

Self-inductance

A time change in the current in a close loop causes the self-induced magnetic flux through the loop to change, and in turns produces an emf in the loop

$$\mathcal{E} = -\frac{d\Phi_{B}}{dt} = -L\frac{di}{dt} \qquad L: \text{self-inductance}$$

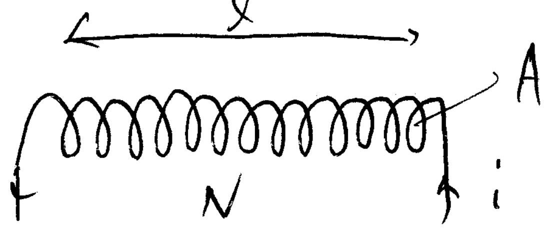
If a close loop consists of N identical turns,

$$L = \frac{N\Phi_B}{i}$$

A circuit component specifically designed to have a well-defined self-inductive effect is an inductor with an inductance L.

Because of the self induction, the current in a close loop cannot change suddenly.

<u>Example</u>: Self-inductance of a solenoid (u, A, N)



$$\Phi_B = A \cdot \mu_0 \cdot \left(\frac{N}{\ell}\right) \cdot i \cdot N$$

$$L = \frac{\Phi_B}{i} = \frac{\mu_0 A \cdot N^2}{\ell}$$

With $\ell = 0.5$ m, $A = 10$ cm^2, $N_1 = 1000$, $L = 2.5$ mH.

- Magnetic energy storage in an inductor:

  Increase an electric current from zero to a finite value I through an inductor, an extra work is done against the self-induced emf.

$$\frac{dW_{extra}}{dt} = i|\mathcal{E}| = i \cdot L \cdot \frac{di}{dt}$$

$$\therefore \ dW_{extra} = d\left(\frac{L}{2} i^2\right)$$

$$W_{extra} = \int_0^I dW_{extra} = \frac{L}{2} I^2$$

This work is not done to push charges through viscous medium, and must lead to magnetic energy increase that is stored somewhere.

$$U_m = \frac{L}{2} I^2 \quad (\text{in an inductor}).$$

<u>Magnetic energy density u_m:</u>

Since the only thing that has changed is the magnetic field inside the inductor (solenoid),

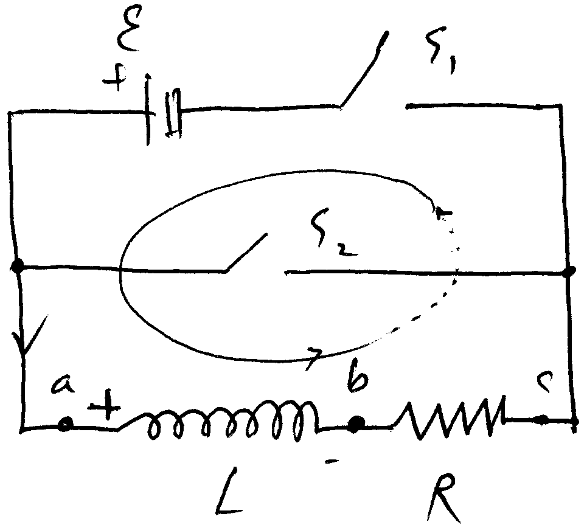
$$U_m = (Al) \cdot u_m = \frac{1}{2} \frac{\mu_0 A \cdot N^2}{l} \cdot I^2 = (Al) \cdot \frac{1}{2\mu_0} (\underbrace{\mu_0 n \cdot I}_{B})^2$$

$$\therefore \ \boxed{u_m = \frac{1}{2\mu_0} B^2}$$

$$\boxed{u_E = \frac{\epsilon_0}{2} E^2}$$

- Resistor-Inductor (R-L) Circuit

Initially, S_1 & S_2 are open. Close S_1, a current starts to build up instead of jumping to $\mathcal{E}/R$, due to the self-induced emf that opposes the change!



By Kirchhoff rule, the electric potential drop along a-b-c-a:

$$\begin{cases} L\dfrac{di}{dt} + iR - \mathcal{E} = 0 \\ i(0) = 0 \end{cases}$$

$$\frac{di}{dt} = -\left(i - \mathcal{E}/R\right)\cdot\frac{R}{L}$$

$$i(t) = \left(1 - e^{-(R/L)\cdot t}\right)\cdot\frac{\mathcal{E}}{R} \qquad i(t=\infty) = \frac{\mathcal{E}}{R}.$$

When S_2 is closed, $I_0 = \mathcal{E}/R$ cannot die out suddenly,

$$L\frac{di}{dt} + iR = 0 \qquad i(t) = I_0\, e^{-(R/L)\cdot t}, \quad \tau_{RL} = \frac{L}{R}$$

Energy dissipation during "current die-down":

$$P(t) = i^2(t)\cdot R = I_0^2 R\, e^{-(2R/L)\cdot t}$$

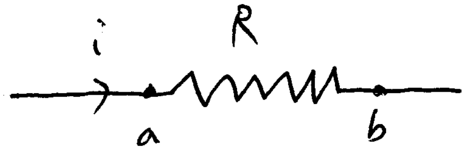
$$\Delta U = \int_0^{+\infty} P(t)\cdot dt = I_0^2 R \cdot \int_0^{+\infty} dt\, e^{-(2R/L)\cdot t} = \frac{L}{2} I_0^2$$

the stored energy inside an inductor (solenoid).

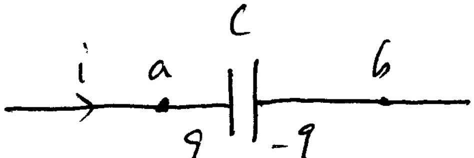
$\boxed{\text{Example 31-7}}$

Kirchhoff Rules for R, C, L:

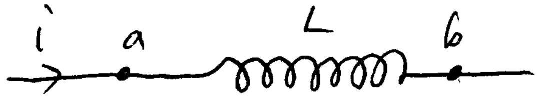
Let a current i flows into any one of these three components, and put q on the first plate of a capacitor that i runs into,



$$V_a - V_b = iR$$



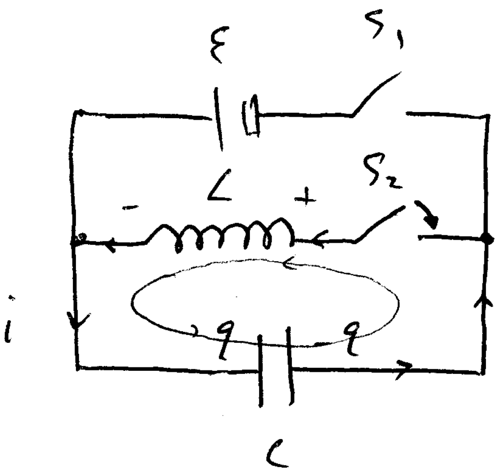
$$V_a - V_b = \frac{q}{C}, \quad i = \frac{dq}{dt}$$



$$V_a - V_b = L\frac{di}{dt}.$$

- L-C Circuit and oscillation

Initially, open S_2, close S_1 until C is charged to $Q_0 = \mathcal{E}C$. Then open S_1. Now close S_2, what happens?



$$\begin{cases} L\dfrac{di}{dt} + \dfrac{q}{C} = 0 \\ i = \dfrac{dq}{dt} \\ q(0) = Q_0, \quad i(0) = 0 \end{cases}$$

$$L\frac{d^2q}{dt} + \frac{q}{C} = 0 \qquad \frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

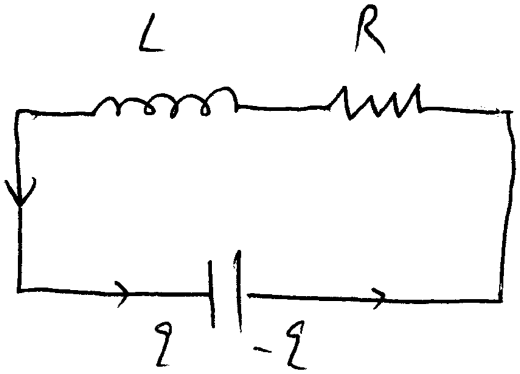
$$q(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \omega_0^2 = \frac{1}{LC}$$

Since $\left.\dfrac{dq}{dt} = i\right|_{t=0} = 0$, $\quad B = 0$

$$\begin{cases} q(t) = Q_0 \cos\left(\dfrac{t}{\sqrt{LC}}\right) \\ i(t) = -\dfrac{Q_0}{\sqrt{LC}} \sin\left(\dfrac{t}{\sqrt{LC}}\right) \end{cases}$$

If the inductor and the rest of the circuit have a small resistance R.

$$\begin{cases} L\dfrac{di}{dt} + iR + \dfrac{q}{c} = 0 \\ i(0) = 0 \\ q(0) = Q_0 \\ \dfrac{dq}{dt} = i \end{cases}$$



Quality factor (Q):

$$Q \equiv \frac{\text{Energy stored}}{\text{Energy loss per oscillation cycle}} \cdot (2\pi) = \frac{\omega L}{R}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = 0$$

Let $q(t) = Ae^{-\lambda t}$, $\quad L\lambda^2 - R\lambda + \frac{1}{c} = 0$

$$\lambda = \frac{R \pm \sqrt{R^2 - 4L/c}}{2L} = \frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

(1) Under-damped oscillation: $\frac{1}{LC} > \left(R/2L\right)^2$ or $\omega_0 > R/2L$

$$q(t) \cong Q_0 e^{-(R/2L)\cdot t} \cdot \cos\left[\left(\omega_0\sqrt{1 - \left(R/2L\omega_0\right)^2}\right)\cdot t\right].$$

(2) Over-damped: $\frac{1}{LC} < \left(R/2L\right)^2$, $\quad q(t) \simeq Q_1 e^{-(R/L)\cdot t} + Q_2 e^{-t/RC}$

Chapt. 32

Maxwell's Equations & Electromagnetic waves

- Electromagnetic waves

Four Maxwell's Equations (really only two are needed) predict that $\vec{E}$ and $\vec{B}$ generate each other, lead to waves that propagate at the speed of light in vacuum, and carry energy here and there, and "activate" electric circuits at a speed only limited by the speed of light

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0} \qquad (1)$$

$$\oint_C \vec{E}\cdot d\vec{\ell} = -\frac{d}{dt}\iint_{S_C} \vec{B}\cdot d\vec{A} \qquad (2)$$

$$\oiint \vec{B}\cdot d\vec{A} = 0 \qquad (3)$$

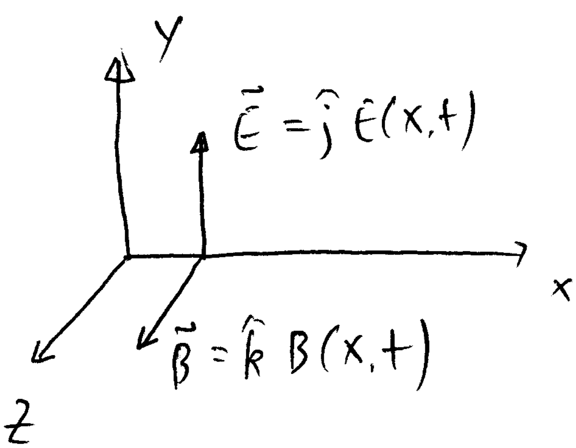
$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 I + \kappa\mu_0\epsilon_0 \frac{d}{dt}\iint_{S_C} \vec{E}\cdot d\vec{A} \qquad (4)$$

In vacuum or media with no free (net) charge Q_{in}, and no current I,

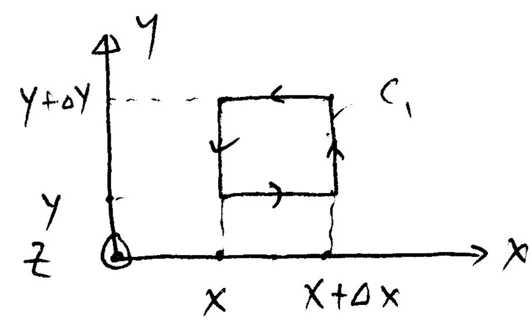
$$\oiint \vec{E}\cdot d\vec{A} = 0 \qquad (1)'$$

$$\oint_C \vec{B}\cdot d\vec{\ell} = \kappa\mu_0\epsilon_0 \frac{d}{dt}\iint_{S_C} \vec{E}\cdot d\vec{A} \qquad (4)'$$

Transverse electric field $\vec{E}$ and transverse magnetic field $\vec{B}$ satisfy wave equations of the form



$$\begin{cases} \dfrac{\partial^2 g}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 g}{\partial t^2} \\ g(x,t) \end{cases}$$

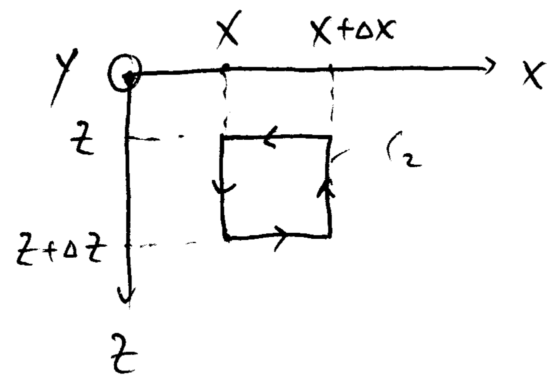


and propagate at the same speed v.

Let $\vec{E} = \hat{j}\, E(x,t)$

$\vec{B} = \hat{z}\, B(x,t)$

(plane-wave)



Integrate $\vec{E}$ along c_1:

$$\oint_{C_1} \vec{E} \cdot d\vec{l} = \left(E(x+\Delta x) - E(x) \right) \cdot \Delta y$$

$$-\iint_{S_{C_1}} \frac{\partial}{\partial t} \vec{B} \cdot d\vec{A} = -\frac{\partial B}{\partial t} \cdot \Delta x \cdot \Delta y$$

$$\boxed{\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}}$$

Integrate $\vec{B}$ along C_2:

$$\oint_{C_2} \vec{B} \cdot d\vec{l} = -\Big(B(x+\Delta x) - B(x) \Big) \Delta z$$

$$\iint_{S_{C_2}} k \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E} \cdot d\vec{A} = k \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \cdot \Delta x \Delta z$$

$$\therefore \boxed{ -\frac{\partial B}{\partial x} = k \mu_0 \epsilon_0 \frac{\partial E}{\partial t} }$$

Now

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial x} \right) = \frac{\partial}{\partial x}\left(-\frac{\partial B}{\partial t} \right) = \frac{\partial}{\partial t}\left(-\frac{\partial B}{\partial x} \right) = k \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$\therefore \boxed{ \frac{\partial^2 E}{\partial x^2} = k \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} } \qquad v = \frac{1}{\sqrt{k}} \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{c}{\sqrt{k}}$$

$$\frac{\partial^2 B}{\partial x^2} = (-) \frac{\partial}{\partial x}\left(-\frac{\partial B}{\partial x} \right) = -k \mu_0 \epsilon_0 \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial t} \right) = -k \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial E}{\partial x} \right)$$

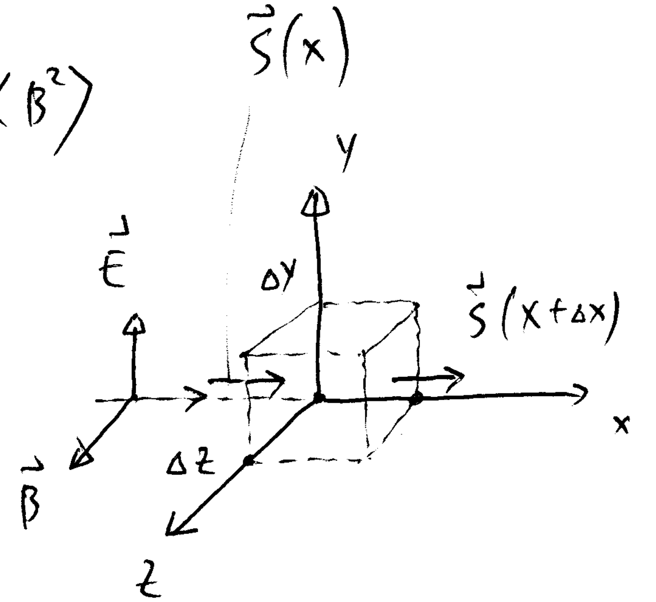
$$= k \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} B$$

$$\therefore \boxed{ \frac{\partial^2 B}{\partial x^2} = k \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} } \qquad v = \frac{c}{\sqrt{k}}, \quad c^2 = \frac{1}{\mu_0 \epsilon_0}$$

- Energy density of an electromagnetic wave and energy flow vector (Poynting vector)

$$u = \frac{k\epsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$$

$$\langle u \rangle = \frac{k\epsilon_0}{2} \langle E^2 \rangle + \frac{1}{2\mu_0} \langle B^2 \rangle$$



An electromagnetic wave with

$$\vec{E} = \hat{j}\, E(x,t)$$

$$\vec{B} = \hat{k}\, B(x,t)$$

propagates along x direction, carrying electromagnetic energy that enters on the left and exits on the right, causing the energy inside to change.

$$\frac{du}{dt} = k\epsilon_0 E \frac{\partial E}{\partial t} + \frac{1}{\mu_0} B \frac{\partial B}{\partial t}$$

$$= -\frac{1}{\mu_0} E \frac{\partial B}{\partial x} - \frac{1}{\mu_0} B \frac{\partial E}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left(\frac{1}{\mu_0} E B\right)$$

$$\frac{d}{dt}\left(\Delta x \Delta y \Delta z \cdot u\right) = (\Delta y \Delta z)\left(\frac{EB}{\mu_0}\bigg|_{x} - \frac{EB}{\mu_0}\bigg|_{x+\Delta x}\right)$$

or

$$\frac{d}{dt}\left(\Delta x \Delta y \Delta z\right) = \left(\Delta y \Delta z \,\hat{i}\right) \cdot \left(\frac{1}{\mu_0} \vec{E} \times \vec{B} \Big|_{x} - \frac{1}{\mu_0} \vec{E} \times \vec{B} \Big|_{x+\Delta x} \right)$$

Energy flow vector (Poynting vector)

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = |\vec{S}| \cdot \hat{S}$$

$$\langle \vec{S} \rangle = \frac{1}{\mu_0} \langle \vec{E} \times \vec{B} \rangle = \langle |\vec{S}| \rangle \cdot \hat{S} = I \cdot \hat{S}$$

Magnitude of Poynting vector: intensity of an e.m. wave, i.e., energy flowing across a unit surface area per unit time

Direction of Poynting vector: direction of the energy flow.

- Harmonic, plane-wave electromagnetic waves

$$\vec{E} = \hat{j}\, E_0 \sin(\omega t - kx)$$

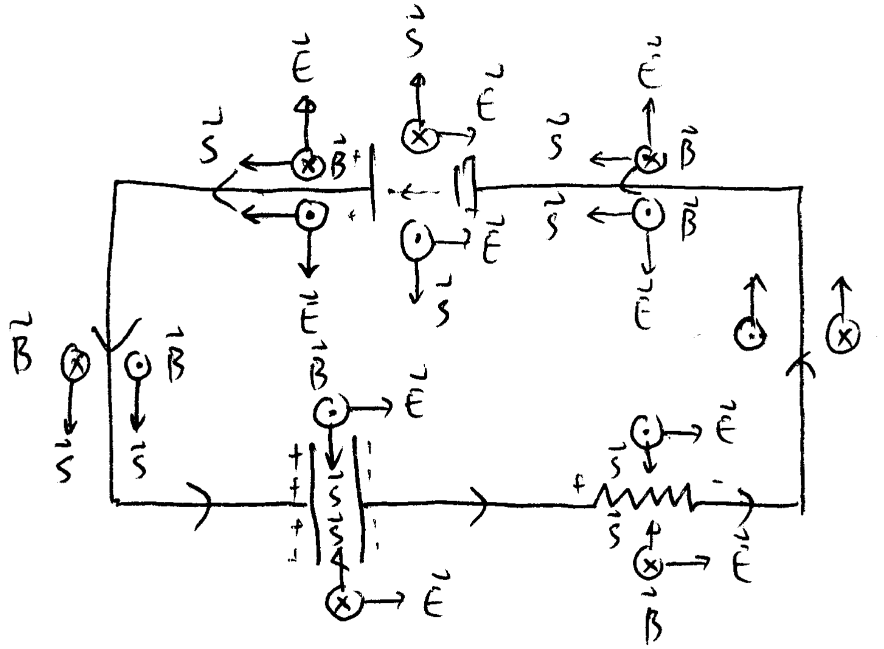
$$\vec{B} = \hat{k}\, B_0 \sin(\omega t - kx)$$

$k = \frac{2\pi}{\lambda}$ is the wavevector

λ is the wavelength

$\omega = \frac{2\pi}{T}$ is the angular freq.

- Energy flow in an electric circuit



Battery (emf) emits energy

R, C (during charging) "absorb" energy

— End of the story —