Exponential Change

Physics 7B - Lecture 9

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Transport Model

- as we have seen, a number of flow phenomena can be described by the general equation

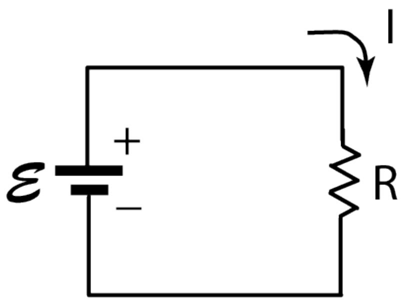
$$I = \Delta V / R$$

flow rate or flux → I

driving potential → ΔV

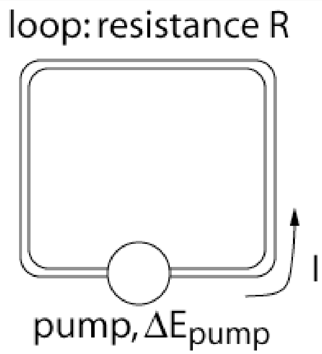
resistance → R

Examples



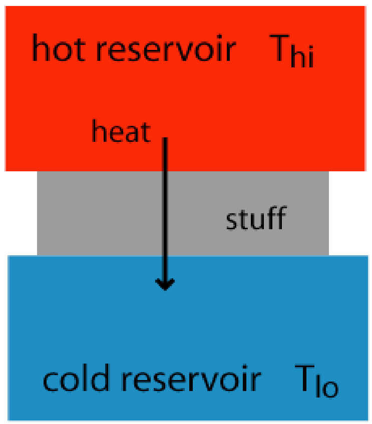
electrical circuit with resistor driven by battery:

$$I = \mathcal{E} / R$$



fluid circuit driven by pump:

$$I = \Delta E_{pump} / R$$



heat transfer between two reservoirs:

$$\mathcal{P} = \Delta T / R$$

Linear Transport - PRS

Which statement is not true about linear transport systems?

A. If you double the driving force (voltage, temperature difference,...) , the current doubles.

B. If you double the resistance, the current is halved.

C. In linear transport systems, energy is being transformed from one type to another.

D. For both fluid flow and electrical circuits, the power dissipated due to the resistance depends linearly on the driving force (pump energy, battery voltage).

E. All of the other statements are true.

Steady State?

- generalized flow (linear transport) equations work in steady state
- what about cases where we are not in a steady state? (changing systems)
- two common kinds of change:

exponential change

oscillations

Exponential Growth

- suppose rate of growth of a population is proportional to the size of the population:

$$\frac{d}{dt}N(t) = kN(t)$$

- N(t) = number in population as a function of time
- k = rate of growth (units: 1/time)

Exponential Growth

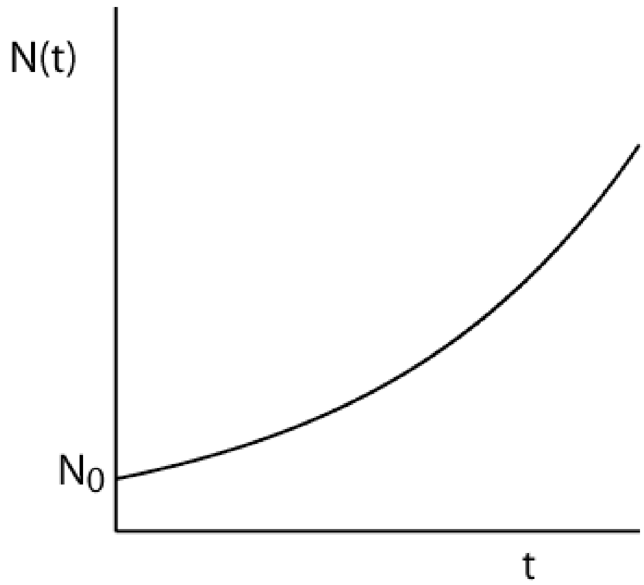
- want to solve equation for N(t)
- need function whose derivative w.r.t. time is that function multiplied by a constant:

$$\begin{aligned} N(t) &= Ce^{kt} \\ \frac{d}{dt}N(t) &= Cke^{kt} = kN(t) \end{aligned}$$

- exactly what we want!

Exponential Growth

- the behavior of N(t) is a rapidly increasing function of time:



$$N(t) \quad = \quad N_0 e^{kt}$$

- N_o is the value of N(t=0)

Examples

- bacteria colony
- interest-bearing fund (APR versus int.)
- global human population?
- spam email :)
- Google stock price
- ?

Exponential Decay

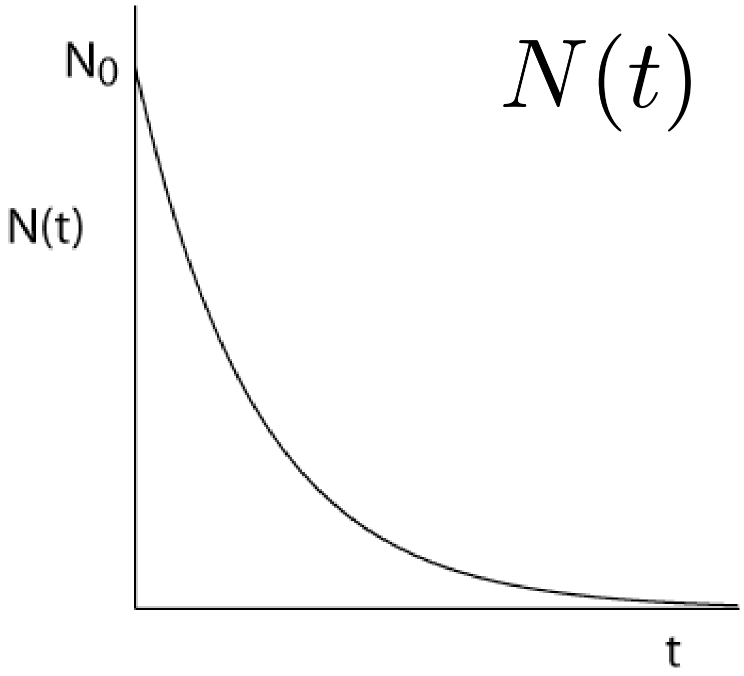
- suppose the rate of disappearance of members of a population is proportional to the number remaining
- this is true, for example, if any member has a random chance k of disappearing in a certain time interval
- the constant k here is the decay rate
- units of k: probability/time

Exponential Decay

- equation is similar to growth case:

$$\frac{d}{dt}N(t) = -kN(t)$$

$$N(t) = N_o e^{-kt}$$



k > 0, but slope of N(t) is <u>always</u> negative!

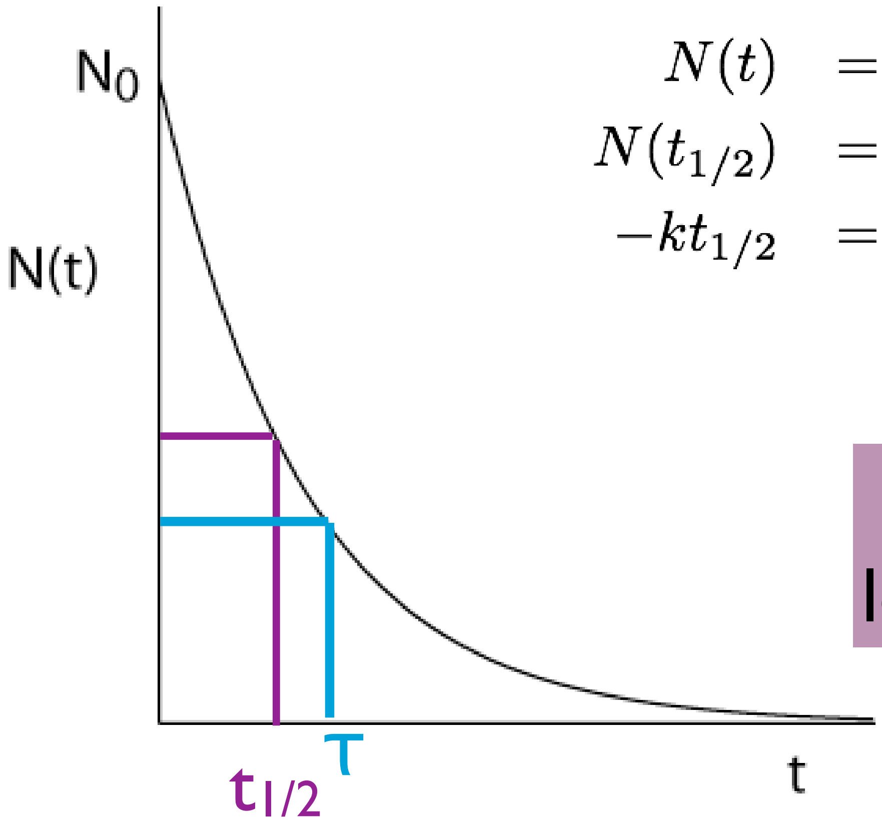
N never gets to zero!

Examples

- radioactive nuclei in atoms
- water leaking from a tank
- charging/discharging capacitor
- approach to terminal velocity

Radioactive Decay

- Suppose you have N atoms of some radioactive isotope
- Suppose the half life of the isotope is 2 days. What is its lifetime (τ=1/k) ?

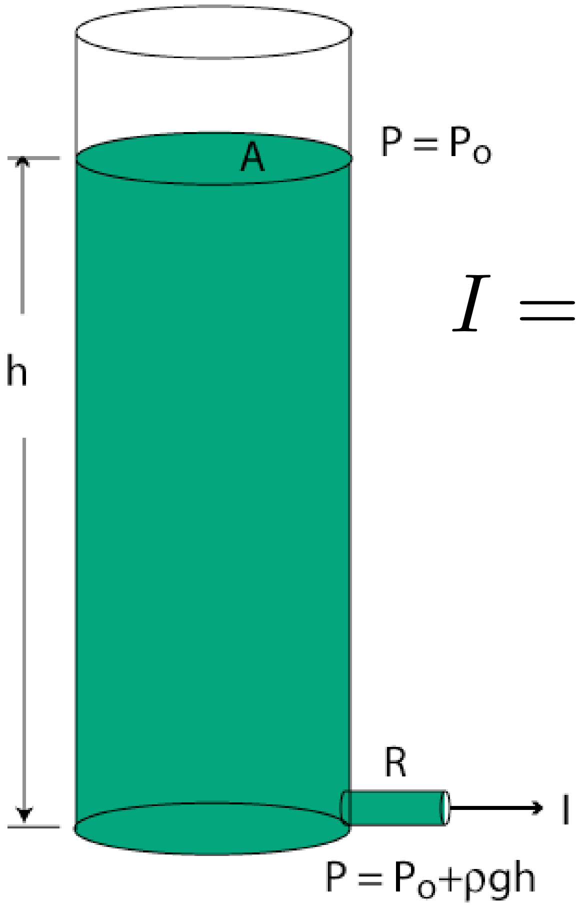


$$
\begin{aligned}
N(t) &= N_0 e^{-kt} \\
N(t_{1/2}) &= N_0/2 \Rightarrow e^{-kt_{1/2}} = 1/2 \\
-kt_{1/2} &= \ln(1/2)
\end{aligned}
$$

After one lifetime we are left with 1/e = 0.368 of N_0

Leaky Bottle

- suppose we have a container with a liquid, and the liquid is getting out



flow rate:

$$I = \Delta P/R \quad = \quad -\rho g h/R = \frac{d(hA)}{dt}$$

Leaky Bottle - PRS

Suppose we observe that we start with a 1-m tall column of liquid. After 1 hour we see that half is gone. The time constant k is ?

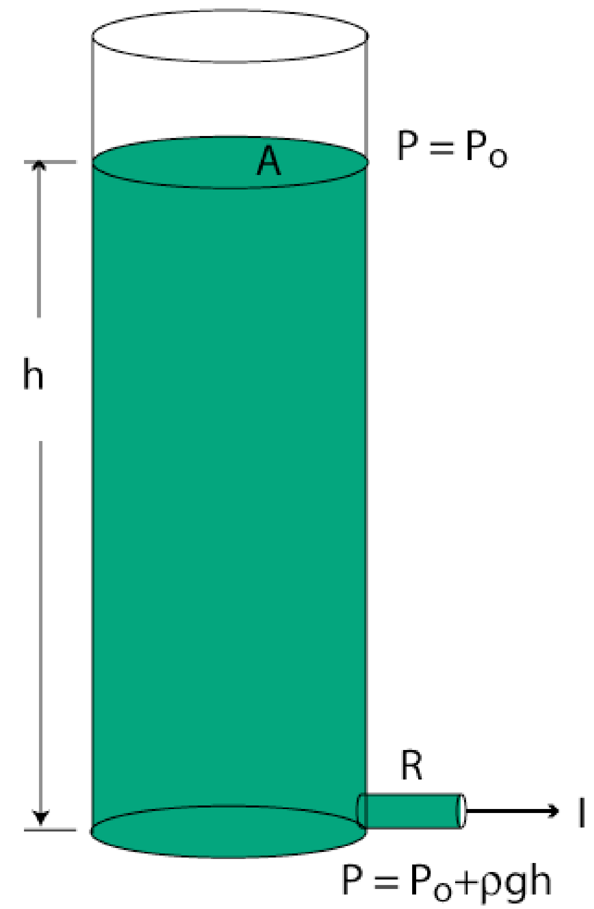
A. $-\ln(1/2)\ \mathrm{hr}^{-1}$

B. $-\ln(2)\ \mathrm{hr}^{-1}$

C. $+\ln(1/2)\ \mathrm{hr}^{-1}$

D. $+\ln(2)\ \mathrm{hr}^{-1}$

E. $-\ln(1/2)\ \mathrm{m}^{-1}$

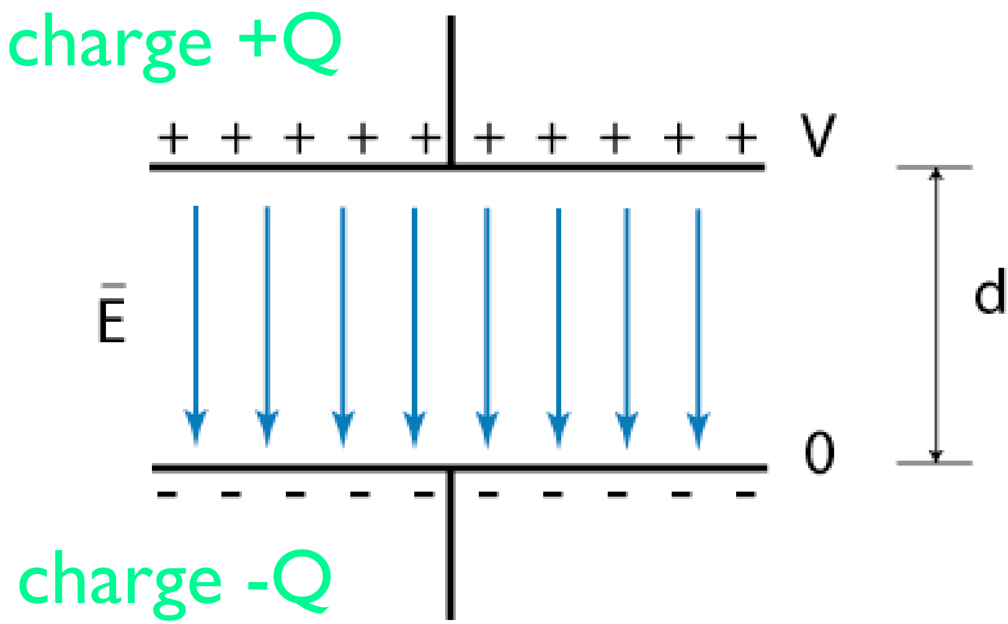


$$h(t) = h_0 e^{-kt}$$

Hint: do the algebra! :)

Capacitors

- a capacitor is a simple device that holds charge, such as two parallel metal plates:

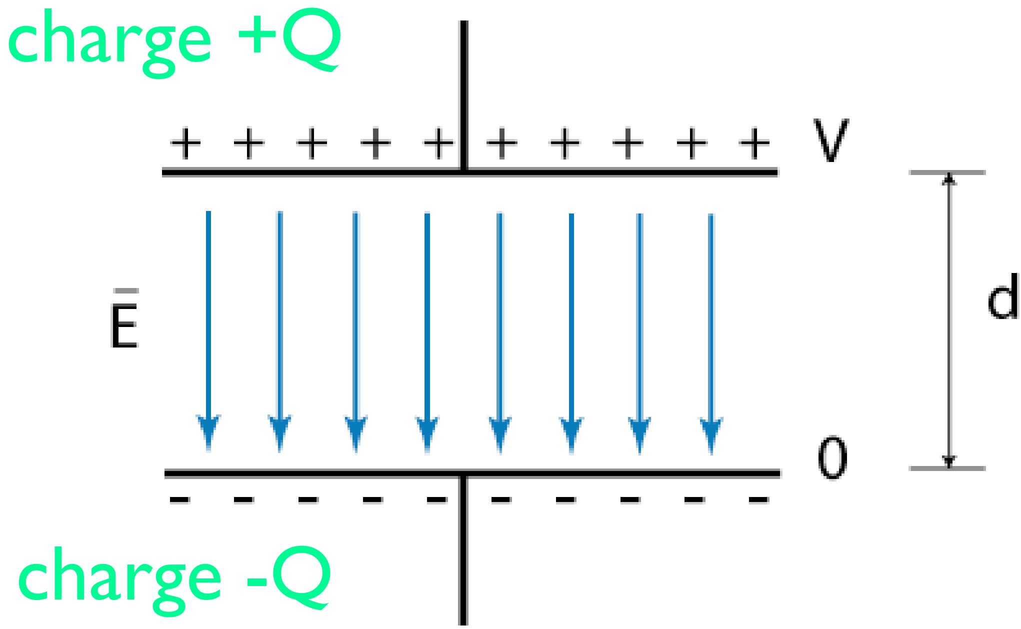


Capacitance is the ratio of the amount of charge on the capacitor to the applied voltage:

$$Q = CV$$

Capacitors

- even though the positive charge on one plate is attracted to the negative charge on the other, the presence of more positive charge on the same plate is repulsive
- it takes energy to charge a capacitor!



energy stored in a capacitor:

$$U = Q^2/2C$$

Capacitors - PRS

Given that $U = Q^2/2C$ and $Q = CV$ for a capacitor, which of the following is true?

A. $U = 1/2\ CV^2$

B. $U = 1/2\ QC^2$

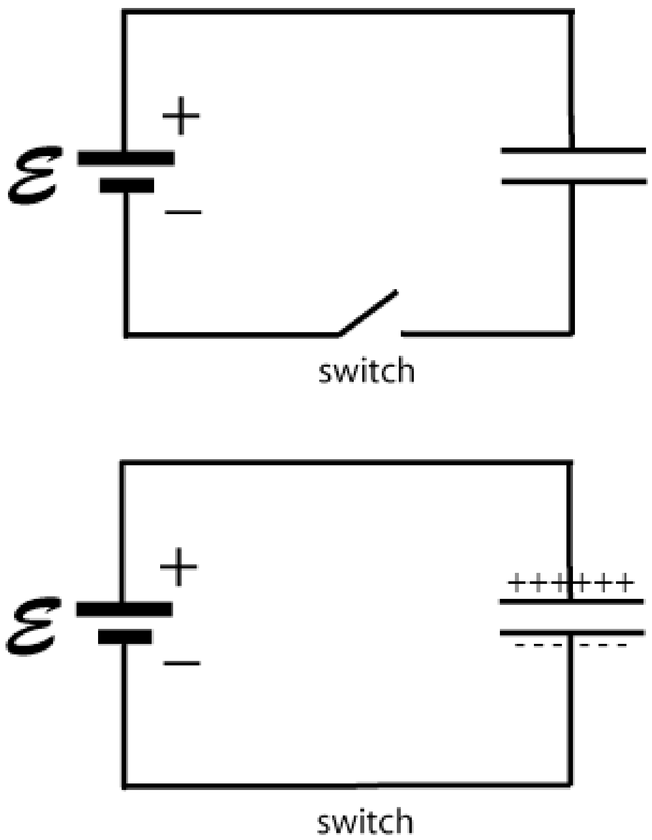
C. $U = C^2/2V$

D. $U = QV$

E. none of the above

Charging a Capacitor

- to charge a capacitor, just connect it to a potential (like a battery):

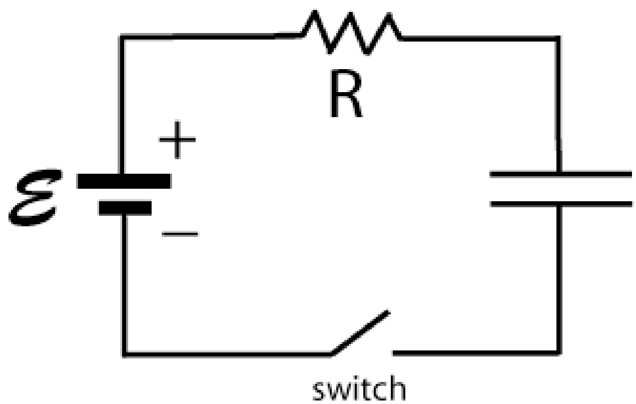


<u>Immediately</u> after closing switch, current flows and charges up capacitor to $Q = C\mathcal{E}$, and then current stops flowing

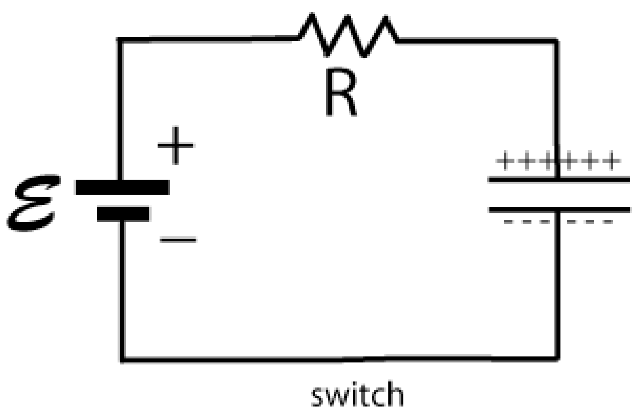
well, not exactly...

Charging a Capacitor

- actually there is always some resistance in the circuit:



When there is resistance, it takes some time for the capacitor to charge.

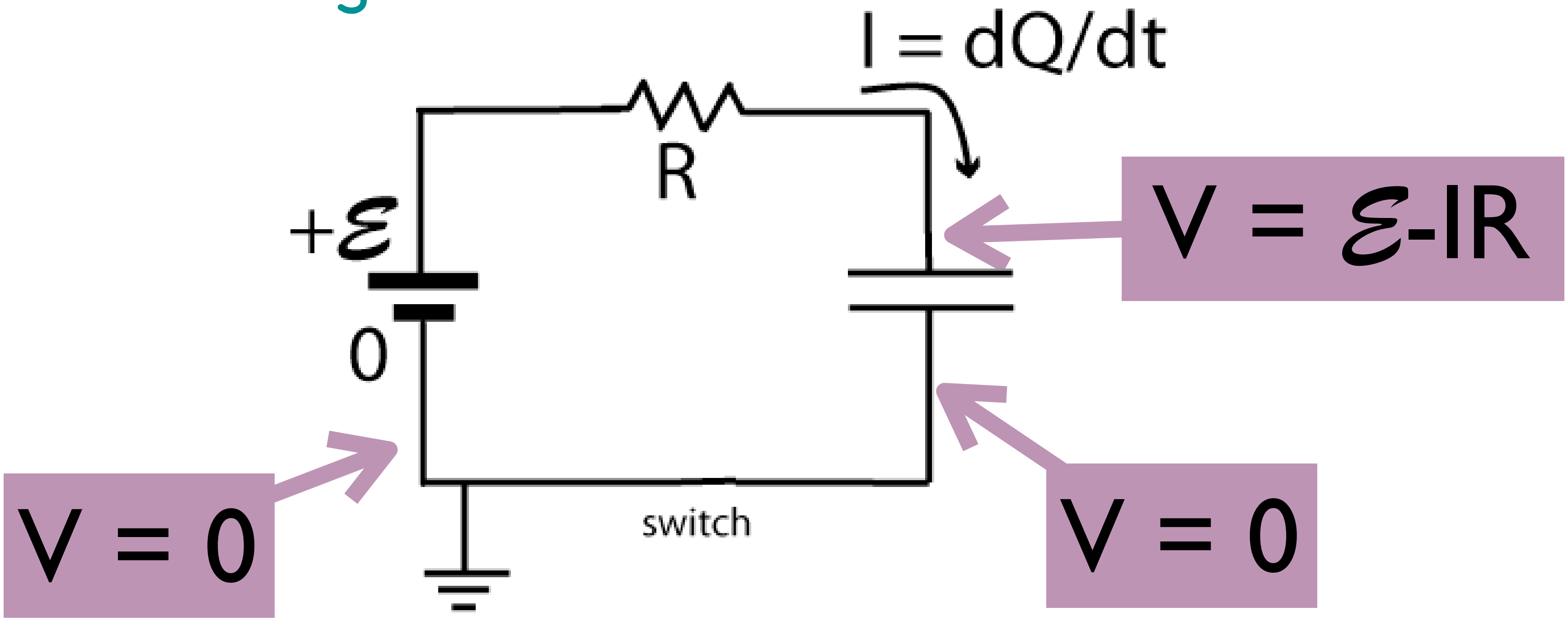


The current is <u>not</u> constant.

We need to analyze this carefully!

Charging a Capacitor

- let's look at the situation just after closing the switch:



But for capacitor, $V = Q/C$!

Charging a Capacitor

- we can write an equation, then, for the current:

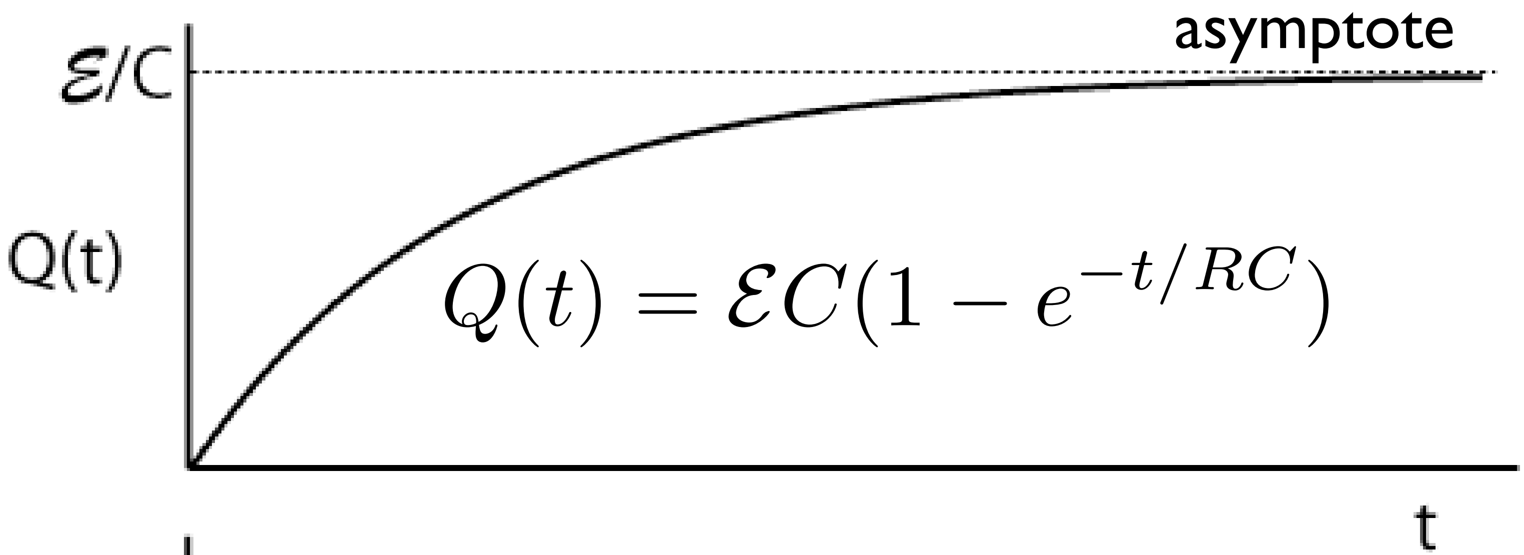
$$\mathcal{E} - R\frac{dQ}{dt} = \frac{Q}{C}$$

$$\frac{\mathcal{E}}{R} - \frac{Q}{RC} = \frac{dQ}{dt}$$

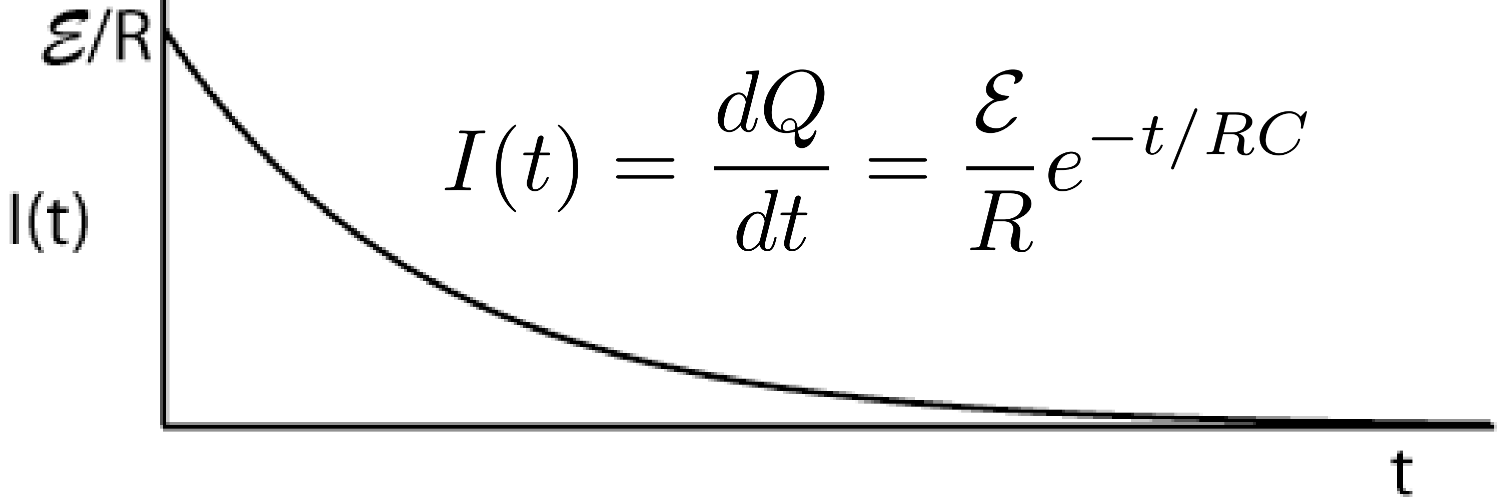
- solution: $$Q(t) = \mathcal{E}C(1 - e^{-t/RC})$$

check that this works!

Charging a Capacitor



$$Q(t) = \mathcal{E}C(1 - e^{-t/RC})$$

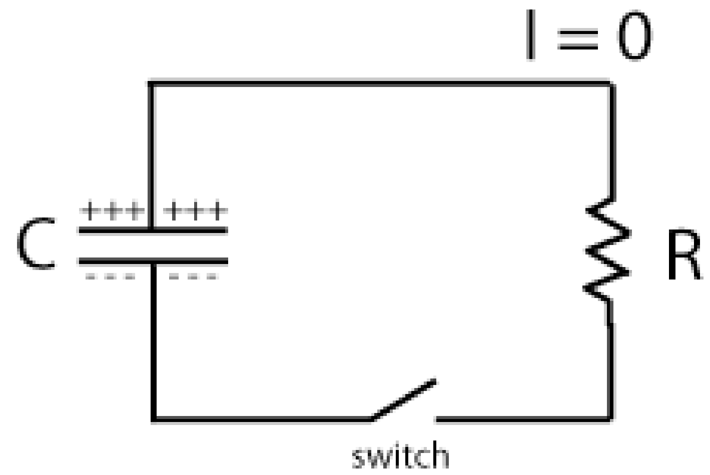


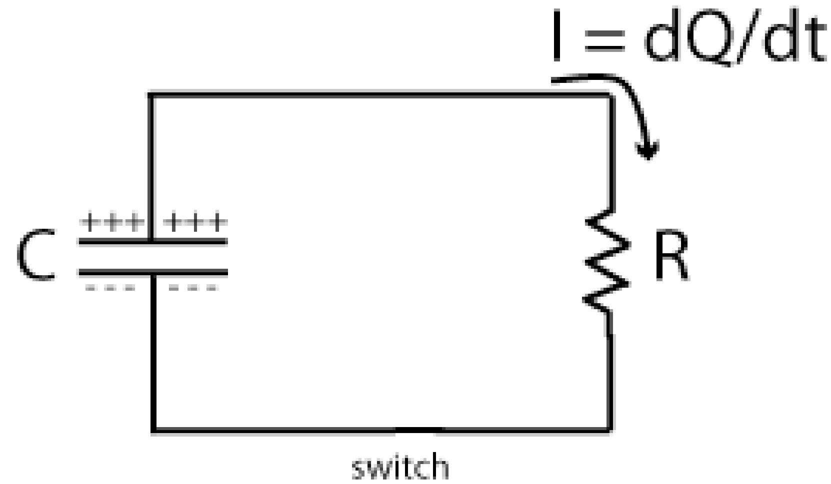
$$I(t) = \frac{dQ}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC}$$

Discharging a Capacitor

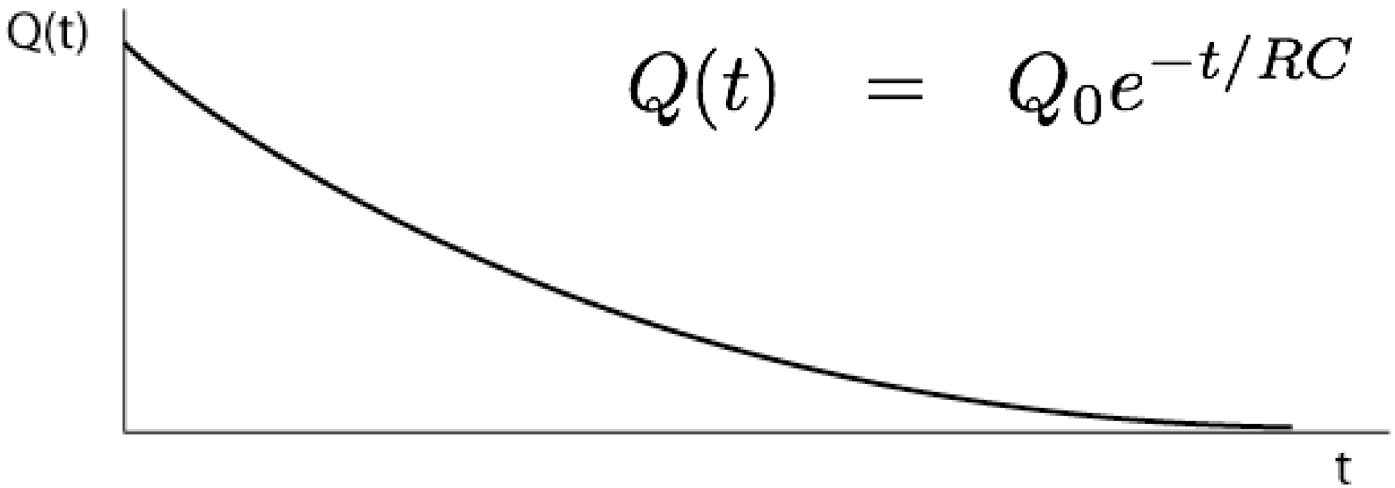
- initially capacitor has charge Q = CV
- no current flows
- then we close the switch
- current flows...initially

  $I = V/R$
- after a while capacitor has no charge, hance no voltage...thus no current!

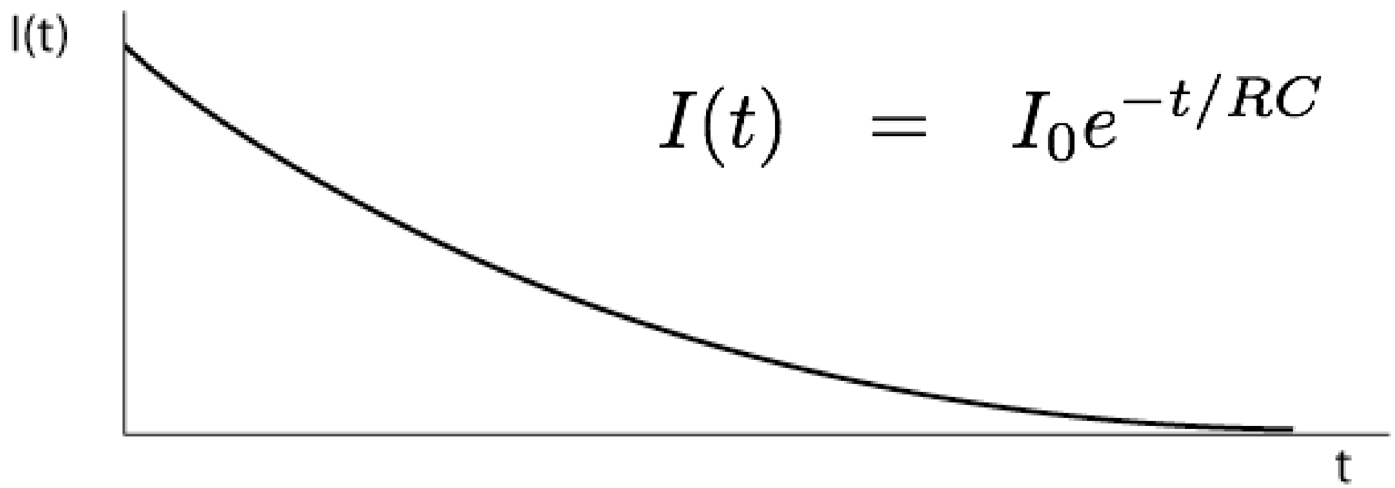




Discharging a Capacitor



$$Q(t) = Q_0 e^{-t/RC}$$



$$I(t) = I_0 e^{-t/RC}$$

Next week: oscillations